

Stationary biharmonic maps from R^m into a Riemannian manifold

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Abstract

We prove that a stationary extrinsic (or intrinsic, respectively) biharmonic map $u \in W^{2,2}(\Omega, N)$ from $\Omega \subset R^m$ into a Riemannian manifold N is smooth away from a closed set of $(m - 4)$ -dimensional Hausdorff measure zero.

§1. Introduction

This is the continuation of [W1,2]. For $m \geq 4$, let $\Omega \subset R^m$ be a bounded domain and $(N, h) \subset R^k$ be a Riemannian manifold. Let $W^{2,2}(\Omega, N)$ be the Sobolev space

$$W^{2,2}(\Omega, N) = \{v \in W^{2,2}(\Omega, R^k) \mid v(x) \in N, \text{ for a.e. } x \in \Omega\}$$

Recall that the hessian energy functional and tension field energy functional are given by

$$H(u) = \int_{\Omega} |\Delta u|^2 dx, \quad T(u) = \int_{\Omega} |(\Delta u)^T|^2 dx, \quad u \in W^{2,2}(\Omega, N)$$

where $(\Delta u)^T$ is the component of Δu tangent to N at u , which is called the tension field in the theory of harmonic maps (see, [EL]). Note that $H(\cdot)$ measures the degree of bending of u and $T(\cdot)$ is the L^2 -norm of tension fields of u , vanishing whenever $u \in W^{2,2}(\Omega, N)$ is a harmonic map.

Chang-Wang-Yang [CWY] initiated an analytic study extrinsic biharmonic maps (i.e. critical points of $H(\cdot)$) for the case that $N = S^{k-1} \subset R^k$, see also [K] for discussions on intrinsic biharmonic maps. In [W1], we make some interesting new observations on biharmonic maps into spheres, mainly new rewritings of the biharmonic map equation via the cross product and applications of estimates of Riesz potentials in Morrey spaces, which yield a partial regularity theorem for stationary intrinsic biharmonic maps into spheres in higher dimensions and new proofs for smoothness of both extrinsic and intrinsic biharmonic maps into spheres in four dimension. For $m = 4$, we prove, in [W2], smoothness for any extrinsic (or intrinsic, respectively) biharmonic map into a Riemannian manifold N . Here we are interested in partial regularity issues for suitable biharmonic maps in dimensions $m \geq 5$. Observe that $\frac{x}{|x|} : B^m \rightarrow S^{m-1}$, for $m \geq 5$, is both an intrinsic biharmonic map and a minimizing extrinsic biharmonic map. Hence, biharmonic maps in dimensions great than or equal to five exhibit singularities. Moreover, motivated by the theory of harmonic

maps (cf. [SU], [H], [E], [B], [CWY1]), we consider the class of stationary biharmonic maps for possible regularities. Now we recall

Definition 1. A map $u \in W^{2,2}(\Omega, N)$ is called an extrinsic (or respectively, intrinsic) biharmonic map if it is a critical point of $H(\cdot)$ (or respectively, $T(\cdot)$) over $W^{2,2}(\Omega, N)$.

Definition 2. (a) An extrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ is called stationary, if it is a critical point of $H(\cdot)$ with respect to domain variations, i.e. for any $X \in C_0^1(\Omega, R^m)$,

$$\frac{d}{dt}\Big|_{t=0} \int_{\Omega} |\Delta u_t|^2 dx = 0 \quad (1.1)$$

(b) An intrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ is called stationary, if it is a critical point of $T(\cdot)$ with respect to domain variations, i.e. for any $X \in C_0^1(\Omega, R^m)$,

$$\frac{d}{dt}\Big|_{t=0} \int_{\Omega} |(\Delta u_t)^T|^2 dx = 0 \quad (1.2)$$

where $u_t(x) = u(x + tX(x))$, for $x \in \Omega$.

Our main theorem is

Theorem A. *For $m \geq 5$. Suppose that $u \in W^{2,2}(\Omega, N)$ is a stationary extrinsic (or intrinsic, respectively) biharmonic map. Then there is a closed set $\Sigma \subset \Omega$, with $H^{m-4}(\Sigma) = 0$, such that $u \in C^\infty(\Omega \setminus \Sigma, N)$.*

Note that if $u \in W^{2,2}(\Omega, N)$ minimizes the hessian energy $H(\cdot)$ then one can check easily that u is a stationary extrinsic biharmonic map. Since the existence of minimizing extrinsic biharmonic maps can be obtained by the direct method of calculus of variations, we have

Corollary B. *For $m \geq 5$. If $u \in W^{2,2}(\Omega, N)$ is a minimizing extrinsic biharmonic map. Then there is a closed $\Sigma \subset \Omega$, with $H^{m-4}(\Sigma) = 0$, such that $u \in C^\infty(\Omega \setminus \Sigma, N)$.*

Motivated by the dimension reduction argument of minimizing harmonic maps (cf. [SU]), it is an interesting question to ask whether the singular set Σ in the Corollary B is of Hausdorff dimension at most $m - 5$. In general, this remains to be an open question so far.

In order to briefly outline the proof of theorem A, we first recall the definition of Morrey spaces (cf. [A]).

Definition 3. For an open set $E \subset R^m$, $1 \leq p < \infty$, and $0 < \lambda \leq m$, the Morrey space $M^{p,\lambda}(E)$ is defined by

$$M^{p,\lambda}(E) = \{f \in L^p(E) : \|f\|_{M^{p,\lambda}(E)}^p \equiv \sup_{B_r \subset E} \{r^{\lambda-m} \int_{B_r} |f|^p(y) dy\} < \infty\}$$

It is clear that $M^{p,m} = L^p(E)$. For $1 < p < \infty$, denote $L_*^p(E)$ as the weak*- L^p space and define $M_*^{p,\lambda}(E)$ by

$$M_*^{p,\lambda}(E) = \{f \in L_*^p(E) : \|f\|_{M_*^{p,\lambda}(E)}^p \equiv \sup_{B_r \subset E} \{r^{\lambda-m} \|f\|_{L_*^p(B_r)}^p\} < \infty\}$$

where

$$\|f\|_{L_*^p(B_r)} \equiv \sup_{t>0} t |\{x \in B_r : |f(x)| \geq t\}|^{\frac{1}{p}}$$

is the weak*- L^p norm of f on B_r .

Now we describe the ideas to prove theorem A. The first main step is to establish the decay estimate of biharmonic maps under the smallness condition of suitable Morrey norms (cf. Lemma 4.1 below), which asserts that if

$$\|\nabla u\|_{M^{4,4}(B_2)} + \|\nabla^2 u\|_{M^{2,4}(B_2)} \tag{1.3}$$

is sufficiently small, then there is a radius $\theta \in (0, \frac{1}{2})$ such that

$$\|\nabla u\|_{M_*^{4,4}(B_\theta)} \leq \frac{1}{2} \|\nabla u\|_{M_*^{4,4}(B_1)} \tag{1.4}$$

There are two key ingredients to prove (1.4). The first step is to establish the existence of an adopted Coulomb gauge frame $\{e_\alpha\}_{\alpha=1}^n$ along with u^*TN under the smallness condition (1.3). Since biharmonic maps may not be invariant under totally geodesic isometric transformations, the original enlargement argument of N in totally geodesic isometric embeddings by Hélein [H] [H1] seems unclear to enable us to find an adopted frame along with the image of biharmonic maps to begin with. Here we take another approach as follows. First, we approximate u by smooth maps $u_\epsilon \in C^\infty(B_1, N)$ such that $\|\nabla u_\epsilon\|_{M^{4,4}(B_1)} + \|\nabla^2 u_\epsilon\|_{M^{2,4}(B_1)}$ is small, uniformly with respect to ϵ . Secondly, based on the smoothness of u_ϵ^*TN and the contractibility of the base manifold B_1 , we can find smooth adopted orthonormal frames $\{\bar{e}_\alpha^\epsilon\}_{\alpha=1}^n$ along with u_ϵ^*TN on B_1 . Thirdly, we find that the smallness condition (1.3) for u_ϵ implies that the $M^{2,4}$ -norm on B_1 of the curvature of the connections $\bar{A}_\epsilon = (\langle D\bar{e}_\alpha^\epsilon, \bar{e}_\beta^\epsilon \rangle)$ is so small that we can apply the Coulomb gauge construction in the Morrey space $M^{4,4}$ by Meyer-Rivière [MR] and Tao-Tian [TT] to achieve Coulomb gauge frames $\{e_\alpha^\epsilon\}_{\alpha=1}^n$ along with u_ϵ^*TN with desired Morrey space estimates on the gradients of the connection forms. Finally, taking ϵ into zero, we obtain an adopted Coulomb gauge frame $\{e_\alpha\}_{\alpha=1}^n$ along with u^*TN with the desired estimates in Morrey spaces (see proposition 3.1 below).

The second step is to rewrite the equation of biharmonic maps under the Coulomb gauge frame to the one with better structures and apply the Adams' estimates of Riesz potential in Morrey spaces to establish a decay estimate of $\|\nabla u\|_{M_*^{4,4}(\cdot)}$ under the condition (1.3). Similar ideas were used in [W1] where biharmonic maps into spheres were considered.

The third major step is to show that away from a closed set Σ of $(m - 4)$ -dimensional Hausdorff measure zero, $\|\nabla u\|_{M^{4,4}(B_r(x))} + \|\nabla^2 u\|_{M^{2,4}(B_r(x))}$ tends to zero as r tends to zero. This is based on the monotonicity inequality (5.2) for stationary biharmonic maps, which was already established by Chang-Wang-Yang [CWY] for stationary extrinsic biharmonic maps, and a general weak type estimate of the biharmonic map equation (cf. Lemma 5.4 below) to control the (interior) boundary terms within the monotonicity inequality.

The paper is organized as follows. In §2, we derive an equation of biharmonic maps via moving frames. In §3, we outline the Coulomb gauge construction in Morrey spaces by [MR] [TT] under the smallness condition (1.2). In §4, we indicate the proof of (1.3) under the condition (1.2). In §5, we derive the monotonicity inequality for stationary intrinsic biharmonic maps, establish a weak type estimate for the biharmonic map equation, and then verify that the condition (1.1) is true for H^{m-4} a.e. $x \in \Omega$.

§2 The Euler-Lagrange equation for biharmonic maps.

This section is devoted to the derivation of Euler-Lagrange equation of biharmonic maps both by the standard way and by moving frames.

We first derive the Euler-Lagrange equation for both intrinsic and extrinsic biharmonic maps in the standard way.

To do this, let $\delta > 0$ and denote N_δ as the δ -neighborhood of N in R^k and $\Pi : N_\delta \rightarrow N$ be the nearest point projection map whenever it exists. Then, for sufficiently small δ , Π exists and is smooth. Moreover, for $y \in N$, $P(y) \equiv \nabla \Pi(y) : R^k \rightarrow T_y N$ is the orthogonal projection onto the tangent space $T_y N$ and $P^\perp(y) \equiv Id - P(y) : R^k \rightarrow (T_y N)^\perp$ is the orthogonal projection onto the normal space $(T_y N)^\perp$. Let $B(\cdot)(\cdot, \cdot)$ be the second fundamental form of N in R^k , defined by

$$B(y)(Y, Z) = D_Y P^\perp(y)(Z), \quad \forall y \in N, \quad Y, Z \in T_y N$$

Proposition 2.1. (a) *An extrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ satisfies, in the sense of distributions,*

$$\Delta^2 u \perp T_u N \tag{2.1}$$

i.e.

$$\int_\Omega \langle \Delta u, \Delta(P(u)(\phi)) \rangle = 0, \quad \forall \phi \in C_0^\infty(\Omega, R^k). \tag{2.2}$$

(b) *An intrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ satisfies, in the sense of distributions,*

$$\begin{aligned} & \{ \Delta^2 u - B(u)(\nabla u, \nabla u) \nabla_u B(u)(\nabla u, \nabla u) \\ & + 2 \nabla \cdot (B(u)(\nabla u, \nabla u) B(u)(\nabla u, \cdot)) \} \perp T_u N \end{aligned} \tag{2.3}$$

where $\nabla \cdot$ is the divergence operator on R^m .

Proof. Since the Euler-Lagrange equations for extrinsic biharmonic maps are easier to derive, we, for simplicity, only outline the proof of (2.3).

For any $\phi \in C_0^\infty(\Omega, R^k)$ and $0 \leq t \ll 1$, set $u_t = \Phi(u + t\phi)$. Then, we have

$$\frac{d}{dt} \Big|_{t=0} \int_{\Omega} |(\Delta u_t)^T|^2 dx = 0$$

Since $\frac{du_t}{dt} \Big|_{t=0} = P(u)(\phi) \in T_u(N)$ and

$$|(\Delta u_t)^T|^2 = |\Delta u_t - B(u_t)(\nabla u_t, \nabla u_t)|^2 = |\Delta u_t|^2 - |B(u_t)(\nabla u_t, \nabla u_t)|^2$$

we have

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \int_{\Omega} |(\Delta u_t)^T|^2 \\ &= \frac{d}{dt} \Big|_{t=0} \int_{\Omega} (|\Delta u_t|^2 - |B(u_t)(\nabla u_t, \nabla u_t)|^2) \\ &= 2 \int_{\Omega} \langle \Delta u, \Delta(P(u)(\phi)) \rangle \\ &\quad - B(u)(\nabla u, \nabla u)[\nabla_u B(u)(\nabla u, \nabla u)(P(u)(\phi)) + 2B(u)(\nabla u, \nabla(P(u)(\phi)))] \end{aligned}$$

this clearly implies the eqn. (2.3) holds in the sense of distributions. ■

An alternative way to express the biharmonic map equations (2.1) and (2.3) by using the projection map P is follows, which will be needed to prove Lemma 5.4.

Proposition 2.2. (a) *An extrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ satisfies, in the sense of distributions,*

$$\Delta^2 u = \Delta(B(u)(\nabla u, \nabla u)) + 2\nabla \cdot (\langle \Delta u, \nabla(P(u)) \rangle) - \langle \Delta(P(u)), \Delta u \rangle \quad (2.4)$$

(b) *An intrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ satisfies, in the sense of distributions,*

$$\begin{aligned} \Delta^2 u &= \Delta(B(u)(\nabla u, \nabla u)) + 2\nabla \cdot (\langle \nabla(P(u)), \Delta u \rangle) - \langle \Delta(P(u)), \Delta u \rangle \\ &\quad + P(u)[B(u)(\nabla u, \nabla u)\nabla_u B(\nabla u, \nabla u) + 2B(u)(\nabla u, \nabla u)B(u)(\nabla u, \nabla(P(u)))] \end{aligned} \quad (2.5)$$

Proof. For simplicity again, we only indicate the proof of (2.5). Suppose that u is an intrinsic biharmonic map. Then (2.3) holds. Note that (2.3) is equivalent to

$$P(u)(\Delta^2 u) = P(u)[B(u)(\nabla u, \nabla u)\nabla_u B(\nabla u, \nabla u) + 2B(u)(\nabla u, \nabla u)B(u)(\nabla u, \nabla(P(u)))]$$

On the other hand, by the product rule, we have

$$\begin{aligned} P(u)(\Delta^2 u) &= \Delta(P(u)(\Delta u)) - \langle \Delta(P(u)), \Delta u \rangle - 2\langle \nabla(P(u)), \nabla \Delta u \rangle \\ &= \Delta^2 u - \Delta(B(u)(\nabla u, \nabla u)) + \langle \Delta(P(u)), \Delta u \rangle - 2\nabla \cdot (\langle \nabla(P(u)), \Delta u \rangle) \end{aligned}$$

Hence, we obtain (2.5). ■

Now we derive an equivalent version of proposition 2.1, 2.2 by using moving frames. For this purpose, we first recall that for an open subset $E \subset \Omega$ and a map $u \in W^{2,2}(E, N)$, denote u^*TN as the pull-back bundle of TN by the map u over E , and call $\{e_\alpha\}_{\alpha=1}^n$, $n = \dim(N)$, as an adopted orthonormal frame along with u^*TN if for a.e. $x \in E$, $\{e_\alpha(x)\}_{\alpha=1}^n$ forms an orthonormal base of $T_{u(x)}N$.

Proposition 2.3. *Let $u \in W^{2,2}(\Omega, N)$ be given and $\{e_\alpha\}_{\alpha=1}^n$ is an adopted orthonormal frame along with u^*TN . Then*

(i) *If u is an intrinsic biharmonic map. Then u satisfies, for $1 \leq \alpha \leq n$,*

$$\begin{aligned} \Delta(\nabla \cdot \langle \nabla u, e_\alpha \rangle) &= 2\langle B(u)(\nabla u, \nabla u), B(u)(\nabla u, \nabla e_\alpha) \rangle \\ &\quad + \langle B(u)(\nabla u, \nabla u), \nabla_u B(u)(\nabla u, \nabla u)(e_\alpha) \rangle + \Delta \langle \nabla u, \nabla e_\alpha \rangle \\ &\quad + \sum_{\beta} (2\nabla \cdot (\langle \Delta u, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle) - \langle \Delta u, e_\beta \rangle \nabla \cdot \langle \nabla e_\alpha, e_\beta \rangle) \\ &\quad - \sum_{\beta} \langle (\Delta u)^T, \nabla e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle - \sum_{\beta} \langle \Delta u, e_\beta \rangle \langle B(u)(e_\alpha, \nabla u), \nabla e_\beta \rangle \\ &\quad + \langle \nabla(B(u)(\nabla u, \nabla u)), \nabla e_\alpha \rangle + \nabla \cdot \langle B(u)(\nabla u, \nabla u), \nabla e_\alpha \rangle \end{aligned} \quad (2.6)$$

(ii) *or an extrinsic biharmonic map. Then u satisfies, for $1 \leq \alpha \leq n$,*

$$\begin{aligned} \Delta \nabla \cdot \langle \nabla u, e_\alpha \rangle &= \Delta \langle \nabla u, \nabla e_\alpha \rangle + \sum_{\beta} (2\nabla \cdot (\langle \Delta u, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle) - \langle \Delta u, e_\beta \rangle \nabla \cdot \langle \nabla e_\alpha, e_\beta \rangle) \\ &\quad - \sum_{\beta} \langle (\Delta u)^T, \nabla e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle - \sum_{\beta} \langle \Delta u, e_\beta \rangle \langle B(u)(e_\alpha, \nabla u), \nabla e_\beta \rangle \\ &\quad + \langle \nabla(B(u)(\nabla u, \nabla u)), \nabla e_\alpha \rangle + \nabla \cdot \langle B(u)(\nabla u, \nabla u), \nabla e_\alpha \rangle \end{aligned} \quad (2.7)$$

Proof. Since the equation of an extrinsic biharmonic map is slightly easier to derive than that of an intrinsic biharmonic map, we will only sketch the proof of (2.7). First, we have, by the eqn. (2.3), that, for $1 \leq \alpha \leq n$,

$$\begin{aligned} \langle \Delta^2 u, e_\alpha \rangle &= 2\langle B(u)(\nabla u, \nabla u), B(u)(\nabla u, \nabla e_\alpha) \rangle \\ &\quad + \langle B(u)(\nabla u, \nabla u), \nabla_u B(u)(\nabla u, \nabla u)(e_\alpha) \rangle \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \Delta^2 u, e_\alpha \rangle &= \Delta \langle \Delta u, e_\alpha \rangle - \nabla \cdot \langle \Delta u, \nabla e_\alpha \rangle - \langle \nabla \Delta u, \nabla e_\alpha \rangle \\ &= \Delta(\nabla \cdot \langle \nabla u, e_\alpha \rangle) - \Delta \langle \nabla u, \nabla e_\alpha \rangle - \nabla \cdot \langle \Delta u, \nabla e_\alpha \rangle - \langle \nabla \Delta u, \nabla e_\alpha \rangle\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta(\nabla \cdot \langle \nabla u, e_\alpha \rangle) &= 2\langle B(u)(\nabla u, \nabla u), B(u)(\nabla u, \nabla e_\alpha) \rangle \\ &\quad + \langle B(u)(\nabla u, \nabla u), \nabla_u B(u)(\nabla u, \nabla u)(e_\alpha) \rangle \\ &\quad + \Delta \langle \nabla u, \nabla e_\alpha \rangle + \nabla \cdot \langle \Delta u, \nabla e_\alpha \rangle + \langle \nabla \Delta u, \nabla e_\alpha \rangle\end{aligned}$$

For the term $\nabla \cdot \langle \Delta u, \nabla e_\alpha \rangle$, we have

$$\begin{aligned}\nabla \cdot \langle \Delta u, \nabla e_\alpha \rangle &= \nabla \cdot \langle (\Delta u)^T, \nabla e_\alpha \rangle + \nabla \cdot \langle B(u)(\nabla u, \nabla u), \nabla e_\alpha \rangle \\ &= \sum_{\beta} \nabla \cdot (\langle \Delta u, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle) + \nabla \cdot \langle B(u)(\nabla u, \nabla u), \nabla e_\alpha \rangle\end{aligned}$$

For the term $\langle \nabla \Delta u, \nabla e_\alpha \rangle$, we have

$$\begin{aligned}\langle \nabla \Delta u, \nabla e_\alpha \rangle &= \langle \nabla(\Delta u)^T, \nabla e_\alpha \rangle + \langle \nabla(B(u)(\nabla u, \nabla u)), \nabla e_\alpha \rangle \\ &= \sum_{\beta} \langle \nabla(\Delta u)^T, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle \\ &\quad + \langle \nabla(\Delta u)^T, P^\perp(u)(\nabla e_\alpha) \rangle + \langle \nabla(B(u)(\nabla u, \nabla u)), \nabla e_\alpha \rangle \\ &= \sum_{\beta} (\nabla \cdot (\langle \Delta u, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle) - \langle \Delta u, e_\beta \rangle \nabla \cdot \langle \nabla e_\alpha, e_\beta \rangle) \\ &\quad - \sum_{\beta} \langle (\Delta u)^T, \nabla e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle - \sum_{\beta} \langle \Delta u, e_\beta \rangle \langle B(u)(e_\alpha, \nabla u), \nabla e_\beta \rangle \\ &\quad + \langle \nabla(B(u)(\nabla u, \nabla u)), \nabla e_\alpha \rangle\end{aligned}$$

Putting these together, we obtain (2.7). ■

§3. Constructions of Coulomb gauge frames in Morrey spaces.

This section is devoted to the construction of Coulomb gauge frame along with u^*TN under the smallness condition (1.3).

For any ball $B_1 \subset R^m$, let us first assume that $u \in W^{2,2}(B_1, N)$ is also smooth. Then u^*TN is a smooth vector bundle over the contractible base manifold B_1 so that u^*TN is trivial on B_1 . Therefore, there exists an adopted smooth orthonormal frame $\{\bar{e}_\alpha\}_{\alpha=1}^n$ along with u^*TN on B_1 . Note that the gauge transformation group \mathcal{G} of u^*TN consists of mappings $R : B_1 \rightarrow \mathbf{SO}(n)$. For any $R \in W^{2,2}(B_1, \mathbf{SO}(n))$, we can rotate $\{\bar{e}_\alpha\}$ to obtain

another orthonormal frame $\{e_\alpha = \sum_\beta R^{\alpha\beta} \bar{e}_\beta\}$, for $1 \leq \alpha \leq n$. Let D be the pull-back covariant derivative on u^*TN , we have

$$De_\alpha = \sum_\beta A^{\alpha\beta} e_\beta$$

where $A = (A^{\alpha\beta})$ is a matrix-value one form, i.e. $A^{\alpha\beta} = \sum_{j=1}^m A_j^{\alpha\beta} dx_j$. For $1 \leq j, l \leq m$, the curvature equation of D enters in the commutation relation

$$\begin{aligned} D_j D_l e_\alpha - D_l D_j e_\alpha &= D_j(A_l^{\alpha\beta} e_\beta) - D_l(A_j^{\alpha\beta} e_\beta) \\ &= (\partial_j(A_l^{\alpha\beta}) - \partial_l(A_j^{\alpha\beta}) + A_l^{\alpha\gamma} A_j^{\gamma\beta} - A_j^{\alpha\gamma} A_l^{\gamma\beta}) e_\beta \\ &= F_{jl}^{\alpha\beta} e_\beta \end{aligned} \quad (3.1)$$

Or

$$\partial_j A_l - \partial_l A_j + [A_j, A_l] = F_{jl} = \mathcal{R}^N(\partial_j u, \partial_l u) \quad (3.2)$$

for short, where $A_j = (A_j^{\alpha\beta})$ and \mathcal{R}^N is the curvature tensor of TN . Here we have used the formula

$$\begin{aligned} D_j D_l e_\alpha - D_l D_j e_\alpha &= \sum_{\beta, \delta} \left\langle \frac{\partial u}{\partial x_j}, e_\beta \right\rangle \left\langle \frac{\partial u}{\partial x_l}, e_\delta \right\rangle u^* (D_{e_\beta}^N D_{e_\delta}^N e_\alpha - D_{e_\delta}^N D_{e_\beta}^N e_\alpha - D_{[e_\beta, e_\delta]}^N e_\alpha) \\ &= \sum_{\beta, \delta} \left\langle \frac{\partial u}{\partial x_j}, e_\beta \right\rangle \left\langle \frac{\partial u}{\partial x_l}, e_\delta \right\rangle u^* (\mathcal{R}_{e_\beta, e_\delta}^N(e_\alpha)) \end{aligned} \quad (3.3)$$

where D^N denotes the Levi-Civita connection on TN . For any $R \in W^{2,2}(B_1, \mathbf{SO}(n))$, the connection form $\bar{A} = (\langle D\bar{e}_\alpha, \bar{e}_\beta \rangle)$ of the frame $\{\bar{e}_\alpha\}$ and the connection form $A = (\langle De_\alpha, e_\beta \rangle)$ of the frame $\{e_\alpha = \sum_\beta R^{\alpha\beta} \bar{e}_\beta\}$ is related by

$$A = R^{-1} dR + R^{-1} \bar{A} R \quad (3.4)$$

and the curvature $|F(\bar{A})| = |F(A)|$ so that the L^2 -norm of curvatures

$$\int_{B_1} |F(A)|^2 = \int_{B_1} |F(\bar{A})|^2$$

is invariant under gauge transformations. Moreover, we observe that (3.3) implies that for a.e. $x \in B_1$,

$$|F(\bar{A})|(x) = |F(A)|(x) \leq C \|\mathcal{R}^N\|_{L^\infty} |\nabla u|^2(x) \leq C_N |\nabla u|^2(x) \quad (3.5)$$

so that

$$\int_{B_1} |F(A)|^2 \leq C \int_{B_1} |\nabla u|^4$$

Based on the above explanations and smoothing approximations for $u \in W^{1,4}(B_1, N)$ satisfying the smallness condition (1.3) by smooth maps $u_\epsilon \in C^\infty(B_1, N)$ preserving the condition (1.3), we can adopt the Coulomb gauge construction in Morrey spaces by [MR] and [TT], which is an extension of the classical Uhlenbeck Coulomb gauge construction in dimension four, to our settings.

Proposition 3.1. *For a ball $B_2 \subset R^m$ of radius 2 and $u \in W^{2,2}(B_2, N)$. Then there exists an $\epsilon_0 > 0$ such that if*

$$\|\nabla u\|_{M^{4,4}(B_2)} \leq \epsilon_0 \quad (3.6)$$

*then there exists an adopted Coulomb gauge orthonormal frame $\{e_\alpha\}_{\alpha=1}^n \subset W^{1,4}(B_1)$, along with u^*TN such that its connection form A satisfies*

$$d^*A = 0, \text{ in } B_1, \quad x \cdot A = 0, \text{ on } \partial B_1 \quad (3.7)$$

$$\|A\|_{M^{4,4}(B_1)} + \|\nabla A\|_{M^{2,4}(B_1)} \leq C\|F(A)\|_{M^{2,4}(B_1)} \leq C\|\nabla u\|_{M^{4,4}(B_1)}^2 \quad (3.8)$$

Proof. The idea to prove the existence of Coulomb gauge frames is follows. First, we show that, under the condition (3.6), u can be approximated strongly in $W^{1,4}$ by $u_\epsilon \in C^\infty(B_1, N)$ satisfying (3.6), uniformly with respect to ϵ . Then, we apply [MR] or [TT] to get Coulomb gauge frames $\{e_\alpha^\epsilon\}_{\alpha=1}^n$ along with u_ϵ^*TN . Finally, taking $\epsilon \rightarrow 0$, we prove all conclusions of the proposition.

To do it, let $\phi : R^m \rightarrow R_+$ be a smooth radial mollifying function so that support $(\phi) \subset B_1$ and $\int_{B_1} \phi = 1$. For $0 < \epsilon \leq 1$, let $\phi^\epsilon(x) = \epsilon^{-m} \phi(\frac{x}{\epsilon})$ for $x \in B_1$ and define

$$u^\epsilon(x) = \int_{B_1} \phi^\epsilon(x-y)u(y) dy = \int_{B_1} \phi(y)u(x-\epsilon y) dy, \quad \forall x \in B_1$$

For fixed $\epsilon \in (0, \frac{1}{2})$ and $x \in B_1$, applying a version of the Poincaré inequality to the function $u_{x,\epsilon}(y) \equiv u(x-\epsilon y) : B_1 \rightarrow R^k$, we have

$$\begin{aligned} \int_{B_1} |u^\epsilon(x) - u_{x,\epsilon}(y)|^4 &\leq C \int_{B_1} |\nabla u_{x,\epsilon}|^4(y) \leq C\epsilon^{4-m} \int_{B_\epsilon(x)} |\nabla u|^4 \\ &\leq C\|\nabla u\|_{M^{4,4}(B_2)}^4 \leq C\epsilon_0^4 \end{aligned} \quad (3.9)$$

Therefore, we have that, for any $x \in B_1$ and $\epsilon \in (0, \frac{1}{2})$,

$$\text{dist}(u^\epsilon(x), N) \leq C\epsilon_0$$

so that if we let ϵ_0 be sufficiently small then $u^\epsilon(B_1) \subset N_\delta$, where δ is so small that $\Pi : N_\delta \rightarrow N$ is smooth. Now we define $u_\epsilon(x) = \Pi(u^\epsilon(x)) : B_1 \rightarrow N$, for $0 < \epsilon < \frac{1}{2}$. Then it is clear that u_ϵ is smooth, $u_\epsilon \rightarrow u$ strongly in $W^{1,4}(B_1, N) \cap W^{2,2}(B_1, N)$ as $\epsilon \downarrow 0$, and

$$\sup_{0 < \epsilon < \frac{1}{2}} \|\nabla u_\epsilon\|_{M^{4,4}(B_1)} \leq C\|\nabla u\|_{M^{4,4}(B_2)} \leq C\epsilon_0 \quad (3.10)$$

For $\epsilon \in (0, \frac{1}{2})$, since the smooth pull-back bundles $(u_\epsilon)^*TN$ are trivial over B_1 , we know that there exist smooth adopted orthonormal frames $\{\bar{e}_\alpha^\epsilon\}_{\alpha=1}^n$ along with $(u_\epsilon)^*TN$ over B_1 . Moreover, (3.10), (3.1), (3.2), (3.5) imply that the L^2 -norm of the curvature of the connections $\bar{A}_\epsilon = (\langle D\bar{e}_\alpha^\epsilon, \bar{e}_\beta^\epsilon \rangle)$ satisfies

$$\|F(\bar{A}_\epsilon)\|_{M^{2,4}(B_1)} \leq C\|\nabla u_\epsilon\|_{M^{4,4}(B_1)}^2 \leq C\|\nabla u\|_{M^{4,4}(B_2)}^2 \leq C\epsilon_0^2 \quad (3.11)$$

Since $\{\bar{e}_\alpha^\epsilon\}_{\alpha=1}^n$ are smooth and their connections \bar{A}_ϵ satisfy (3.11), we can apply theorem 4.6 of [TT] (or theorem I.3 of [MR]) to \bar{A}_ϵ to conclude that there is an element $R_\epsilon : B_1 \rightarrow \mathbf{SO}(n)$ satisfying $\nabla R_\epsilon \in M^{4,4}(B_1)$, $\nabla^2 R_\epsilon \in M^{2,4}(B_1)$ such that the connection form $A_\epsilon \equiv R_\epsilon^{-1}dR_\epsilon + R_\epsilon^{-1}\bar{A}_\epsilon R_\epsilon$ of the adopted frame $e_\alpha^\epsilon(x) \equiv \sum_\beta R_\epsilon^{\alpha\beta}\bar{e}_\beta^\epsilon$ satisfies

$$d^*A_\epsilon = 0, \text{ in } B_1, \quad x \cdot A_\epsilon = 0, \text{ on } \partial B_1 \quad (3.12)$$

$$\|A_\epsilon\|_{M^{4,4}(B_1)} + \|\nabla A_\epsilon\|_{M^{2,4}(B_1)} \leq C\|F(\bar{A}_\epsilon)\|_{M^{2,4}(B_1)} \leq C\epsilon_0 \quad (3.13)$$

Note that $\{e_\alpha^\epsilon\}_{\alpha=1}^n$ is an adopted orthonormal frame along with $(u_\epsilon)^*TN$. Now we need to bound $M^{4,4}$ -norm of $\{e_\alpha^\epsilon\}_{\alpha=1}^n$. Since

$$De_\alpha^\epsilon = \sum_\beta \langle De_\alpha^\epsilon, e_\beta^\epsilon \rangle e_\beta^\epsilon + P^\perp(u_\epsilon)(De_\alpha^\epsilon) \quad (3.14)$$

Moreover, since $P^\perp(u_\epsilon)(e_\alpha^\epsilon) = 0$, we have

$$P^\perp(u_\epsilon)(De_\alpha^\epsilon) = -D(P^\perp(u_\epsilon))(e_\alpha^\epsilon) = -B(u_\epsilon)(e_\alpha^\epsilon, \nabla u_\epsilon) \quad (3.15)$$

Therefore

$$|De_\alpha^\epsilon| \leq C(|A_\epsilon| + |Du_\epsilon|) \quad (3.16)$$

This, combined with (3.10) and (3.13), yields

$$\sum_\alpha \|De_\alpha^\epsilon\|_{M^{4,4}(B_1)} \leq C(\|A_\epsilon\|_{M^{4,4}(B_1)} + \|\nabla u_\epsilon\|_{M^{4,4}(B_1)}) \leq C\|\nabla u\|_{M^{4,4}(B_2)} \quad (3.17)$$

Therefore, by choosing a subsequence of $\epsilon \downarrow 0$, we can assume that $e_\alpha^\epsilon \rightarrow e_\alpha$ weakly in $W^{1,4}(B_1)$, strongly in $L^4(B_1)$, and a.e. in B_1 . Since $u_\epsilon \rightarrow u$ strongly in $W^{1,4}(B_1)$, this implies that $\{e_\alpha\}_{\alpha=1}^n \subset W^{1,4}(B_1)$ is an adopted orthonormal frame along with u^*TN . Moreover, the bound of $\|\nabla A_\epsilon\|_{L^2(B_1)}$ implies that $A_\epsilon \rightarrow A \equiv (\langle De_\alpha, e_\beta \rangle)$, the connection form of $\{e_\alpha\}_{\alpha=1}^n$, weakly in $W^{1,2}(B_1)$ and strongly in $L^2(B_1)$. Hence, by taking ϵ into zero, (3.12) and (3.13) imply that A satisfies (3.7) and (3.8). Therefore, the proof of proposition 3.1 is complete. \blacksquare

§4. Decay estimates in Morrey spaces

This section is devoted to the proof of following decay Lemma in Morrey spaces.

Lemma 4.1. *There exist an $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $u \in W^{2,2}(B_2, N)$ is an extrinsic (or intrinsic, respectively) biharmonic map satisfying*

$$\|\nabla u\|_{M^{4,4}(B_2)} + \|\nabla^2 u\|_{M^{2,4}(B_2)} \leq \epsilon_0 \quad (4.1)$$

then

$$\|u\|_{M_*^{4,4}(B_{\theta_0})} \leq \frac{1}{2} \|u\|_{M_*^{4,4}(B_1)} \quad (4.2)$$

In order to prove Lemma 4.1, we need to recall some estimates of Riesz potentials in Morrey spaces, due to Adams [A] (see also Adams-Lewis [AL]), and a sharp version of the interpolation inequality of Nirenberg type (cf. Adams-Frazier [AF]). For this, denote I_α as the Riesz potential operator, i.e. the operator whose convolution kernel is $|x|^{\alpha-m}$, $x \in R^m$. Then proposition 3.1, 3.2, and theorem 3 of [A] can be stated

Proposition 4.2 ([A]). (a) *If $\alpha > 0$, $0 < \lambda \leq m$, $1 < p < \frac{\lambda}{\alpha}$, and $f \in L^p(R^m) \cap M^{p,\lambda}(R^m)$, then*

$$\|I_\alpha(f)\|_{L^{\tilde{p}}(R^m)} \leq C \|f\|_{M^{p,\lambda}(R^m)}^{\frac{\alpha p}{\lambda}} \|f\|_{L^p(R^m)}^{1-\frac{\alpha p}{\lambda}} \quad (4.3)$$

$$\|I_\alpha(f)\|_{M^{\tilde{p},\lambda}(R^m)} \leq C \|f\|_{M^{p,\lambda}(R^m)} \quad (4.4)$$

where $\tilde{p} = \frac{\lambda p}{\lambda - \alpha p}$.

(b) *If $0 < \alpha < \lambda \leq m$, $f \in L^1(R^m) \cap M^{1,\lambda}(R^m)$. Then*

$$\|I_\alpha(f)\|_{L_*^{\frac{\lambda}{\lambda-\alpha}}(R^m)} \leq C \|f\|_{M^{1,\lambda}(R^m)}^{\frac{\alpha}{\lambda}} \|f\|_{L^1(R^m)}^{1-\frac{\alpha}{\lambda}} \quad (4.5)$$

$$\|I_\alpha(f)\|_{M_*^{\frac{\lambda}{\lambda-\alpha},\lambda}(R^m)} \leq C \|f\|_{M^{1,\lambda}(R^m)} \quad (4.6)$$

Proposition 4.3 ([AF]). *For any ball $B_r \subset R^m$. If $f \in W^{2,2}(B_r) \cap BMO(B_r)$. Then*

$$\|\nabla f\|_{L^4(B_r)}^2 \leq C [f]_{BMO(B_r)} (r^{-1} \|\nabla f\|_{L^2(B_r)} + \|\nabla^2 f\|_{L^2(B_r)}) \quad (4.7)$$

where BMO is the BMO space which is defined by: for any open set $E \subset R^m$,

$$BMO(E) = \{f \in L_{loc}^1(E) : [f]_{BMO(E)} \equiv \sup_{B_r \subset E} \{ \inf_{c \in R} r^{-m} \int_{B_r} |f - c| \} < \infty\}$$

Proof. It follows essentially from Lemma 2.2 of [AF], of which a special case implies that if $g \in \text{BMO}(R^m) \cap W^{2,2}(R^m)$ then

$$\|\nabla g\|_{L^4(R^m)}^2 \leq C[g]_{\text{BMO}(R^m)} \|\nabla^2 g\|_{L^2(R^m)} \quad (4.8)$$

Now, for $f \in \text{BMO}(B_r) \cap W^{2,2}(B_r)$, it is easy to see that there exist an extension $\bar{f} \in \text{BMO}(R^m) \cap W^{2,2}(R^m)$ of f such that

$$[\bar{f}]_{\text{BMO}(R^m)} \leq C[f]_{\text{BMO}(B_r)}, \|\nabla^2 \bar{f}\|_{L^2(R^m)} \leq C(r^{-1} \|\nabla f\|_{L^2(B_r)} + \|\nabla^2 f\|_{L^2(B_r)}) \quad (4.9)$$

Therefore, applying (4.8) with g replaced by \bar{f} , we obtain (4.7). \blacksquare

We also need some basic facts on the fundamental solution of Δ^2 on R^m for $m \geq 5$. First, it is easy to check that for a suitably chosen constant c_m , $G(x-y) = c_m |x-y|^{4-m}$ is the fundamental solution of Δ^2 in R^m , i.e. $\Delta^2 G(x-y) = \delta(x-y)$ for $x, y \in R^m$. Now we have

Proposition 4.4. *Let G be as above. Then, for any $x \neq 0 \in R^m$, we have*

$$|\nabla G|(x) \leq C|x|^{3-m}, \quad |\nabla^2 G|(x) \leq C|x|^{2-m}, \quad |\nabla^3 G|(x) \leq C|x|^{1-m} \quad (4.10)$$

and $\nabla^4 G$ is a Calderon-Zygmund kernel in R^m .

Proof. Direct calculations give (4.10). Moreover, $\nabla^4 G(x) = g(\frac{x}{|x|})|x|^{-m}$, for $x \neq 0 \in R^m$, with $g \in C^1(S^{m-1})$ satisfying $\int_{S^{m-1}} g(x) dH^{m-1} = 0$. Hence, $\nabla^4 G$ is a Calderon-Zygmund kernel. \blacksquare

Proof of Lemma 4.1. Since the case of extrinsic biharmonic maps can be handled similarly to, indeed slightly easier than, that of intrinsic biharmonic maps. We will only outline the proof of (4.2) for intrinsic biharmonic maps.

First, let $\epsilon_0 > 0$ be as same as proposition 3.1. It then follows from proposition 3.1 that there is an adopted Coulomb gauge frame $\{e_\alpha\}_{\alpha=1}^n$ along with the bundle u^*TN , such that its connection form $A = (\langle De_\alpha, e_\beta \rangle)$ satisfies

$$\|A\|_{M^{4,4}(B_1)} + \|\nabla A\|_{M^{2,2}(B_1)} \leq C\|F(A)\|_{M^{2,4}(B_1)} \leq C\|\nabla u\|_{M^{4,4}(B_1)}^2 \quad (4.11)$$

Now we let $\tilde{u} \in W^{2,2}(R^m, R^k)$, $\{\tilde{e}_\alpha\}_{\alpha=1}^n, \tilde{A}$ be extensions of $u, \{e_\alpha\}_{\alpha=1}^n, A, B$ to R^m such that $|\tilde{e}_\alpha| \leq 1, \tilde{A}^{\alpha\beta} = \langle D\tilde{e}_\alpha, \tilde{e}_\beta \rangle$, and

$$\|\nabla^2 \tilde{u}\|_{M^{2,4}(R^m)} \leq C\|\nabla^2 u\|_{M^{2,4}(B_1)} \quad (4.12)$$

$$\|\nabla \tilde{u}\|_{M^{4,4}(R^m)} \leq \|\nabla u\|_{M^{4,4}(B_1)} \quad (4.13)$$

$$\|\tilde{A}\|_{M^{4,4}(R^m)} + \|\nabla \tilde{A}\|_{M^{2,m}(R^m)} \leq C\|\nabla u\|_{M^{4,4}(B_1)}^2 \quad (4.14)$$

Also, let \tilde{B} be an extension of B from N to R^k and

$$(\Delta\tilde{u})^T \equiv \Delta\tilde{u} - \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}) \quad (4.15)$$

Now, for $1 \leq \alpha \leq n$, we define nine auxiliary functions ω_i^α , $1 \leq i \leq 9$, as follows.

$$\begin{aligned} \omega_1^\alpha(x) &= 2 \int_{R^m} G(x-y) \langle \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}), \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{e}_\alpha) \rangle(y) dy \\ \omega_2^\alpha(x) &= \int_{R^m} G(x-y) \langle \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}), \nabla_{\tilde{u}} \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u})(e_\alpha) \rangle(y) dy \\ \omega_3^\alpha(x) &= \int_{R^m} G(x-y) \Delta \langle \nabla\tilde{u}, \nabla\tilde{e}_\alpha \rangle(y) dy \\ \omega_4^\alpha(x) &= 2 \sum_{\beta} \int_{R^m} G(x-y) \nabla \langle \langle \Delta\tilde{u}, \tilde{e}_\beta \rangle \langle \nabla\tilde{e}_\alpha, \tilde{e}_\beta \rangle \rangle(y) dy \\ \omega_5^\alpha(x) &= - \sum_{\beta} \int_{R^m} \langle \Delta\tilde{u}, \tilde{e}_\beta \rangle \nabla \cdot \langle \nabla\tilde{e}_\alpha, \tilde{e}_\beta \rangle \rangle(y) dy \\ \omega_6^\alpha(x) &= - \sum_{\beta} \int_{R^m} G(x-y) \langle (\Delta\tilde{u})^T, \nabla\tilde{e}_\beta \rangle \langle \nabla\tilde{e}_\alpha, \tilde{e}_\beta \rangle(y) dy \\ \omega_7^\alpha(x) &= - \sum_{\beta} \int_{R^m} G(x-y) \langle \Delta\tilde{u}, \tilde{e}_\beta \rangle \langle \tilde{B}(\tilde{u})(\tilde{e}_\alpha, \nabla\tilde{u}), \nabla\tilde{e}_\beta \rangle(y) dy \\ \omega_8^\alpha(x) &= \int_{R^m} G(x-y) \langle \nabla(\tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u})), \nabla\tilde{e}_\alpha \rangle(y) dy \\ \omega_9^\alpha(x) &= \int_{R^m} G(x-y) \nabla \cdot \langle \tilde{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}), \nabla\tilde{e}_\alpha \rangle \end{aligned}$$

Now we want to estimate $\|\nabla^2 \omega_i^\alpha\|_{M_*^{2,4}(R^m)}$, for $1 \leq i \leq 9$.

Lemma 4.5. *For $1 \leq i \leq 9$, $\nabla \omega_i^\alpha \in M_*^{4,4}(R^m)$ and*

$$\|\nabla \omega_i\|_{M_*^{4,4}(R^m)} \leq C \|\nabla u\|_{M^{4,4}(B_1)}^2 \quad (4.16)$$

Proof of Lemma 4.5. It is easy to see that, by (4.10),

$$|\nabla \omega_2|(x) \leq C \int_{R^m} |\nabla G(x-y)| |\nabla \tilde{u}|^4(y) dy = CI_3(|\nabla \tilde{u}|^4)(x)$$

and

$$|\nabla \omega_1|(x) \leq C \int_{R^m} |\nabla G(x-y)| |\nabla \tilde{u}|^3 |\nabla \tilde{e}_\alpha|(y) dy = CI_3(|\nabla \tilde{u}|^3 |\nabla \tilde{e}_\alpha|)(x)$$

Since $|\nabla\tilde{u}|^4, |\nabla\tilde{u}|^3|\nabla\tilde{e}_\alpha| \in M^{1,4}(R^m)$, (4.6) implies that $\nabla\omega_i^\alpha \in M_*^{4,4}(R^m)$ for $i = 1, 2$, and

$$\begin{aligned}\|\nabla\omega_2^\alpha\|_{M_*^{4,4}(R^m)} &\leq C\|\nabla\tilde{u}\|_{M^{1,4}(R^m)}^4 \\ &\leq C\|\nabla\tilde{u}\|_{M^{4,4}(R^m)}^4 \\ &\leq C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)}^2\end{aligned}$$

Similarly, for $\nabla\omega_1^\alpha$, we have, by Hölder inequality,

$$\begin{aligned}\|\nabla\omega_1^\alpha\|_{M_*^{4,4}(R^m)} &\leq C\|\nabla\tilde{u}\|^3\|\nabla\tilde{e}_\alpha\|_{M^{1,4}(R^m)} \\ &\leq C\|\nabla\tilde{u}\|_{M^{4,4}(R^m)}^3\|\nabla\tilde{e}_\alpha\|_{M^{4,4}(R^m)} \\ &\leq C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)}^2\end{aligned}$$

Here we have used the fact that

$$|\nabla\tilde{e}_\alpha| \leq C(|\tilde{A}| + |\nabla\tilde{u}|), \quad (4.17)$$

(4.11), (4.12), and (4.14) in the last step. For ω_3^α , we have

$$\begin{aligned}|\nabla\omega_3^\alpha|(x) &\leq C\int_{R^m} |\nabla^3 G(x-y)| |\nabla\tilde{u}| |\nabla\tilde{e}_\alpha|(y) dy \\ &= CI_1(|\nabla\tilde{u}| |\nabla\tilde{e}_\alpha|)(x)\end{aligned}$$

Therefore, (4.4) and the Hölder inequality imply that $\nabla\omega_3^\alpha \in M^{4,4}(R^m)$ and

$$\begin{aligned}\|\nabla\omega_3^\alpha\|_{M^{4,4}(R^m)} &\leq C\|\nabla\tilde{u}\| \|\nabla\tilde{e}_\alpha\|_{M^{4,4}(R^m)} \\ &\leq C\|\nabla\tilde{u}\|_{M^{4,4}(R^m)} \|\nabla\tilde{e}_\alpha\|_{M^{4,4}(R^m)} \\ &\leq C\|\nabla u\|_{M^{4,4}(B_1)}\end{aligned} \quad (4.18)$$

where we have used (4.11), (4.12), (4.14), (4.16), and (4.17) in the last step. Since $M^{4,4}(R^m) \subset M_*^{4,4}(R^m)$ and

$$\|f\|_{M_*^{4,4}(R^m)} \leq C\|f\|_{M^{4,4}(R^m)},$$

(4.18) yields that (4.16) holds for $i = 3$.

For ω_4^α , since

$$\begin{aligned}|\nabla\omega_4^\alpha|(x) &\leq C\int_{R^m} |\nabla^2 G|(x-y) |\Delta\tilde{u}| |\tilde{A}|(y) dy \\ &= CI_2(|\Delta\tilde{u}| |\tilde{A}|)(x)\end{aligned}$$

Hence, applying (4.4) with $p = \frac{4}{3}$, $\lambda = 4$, and $\alpha = 2$, we have that $\nabla\omega_4^\alpha \in M^{4,4}(R^m)$ and

$$\begin{aligned} \|\nabla\omega_4^\alpha\|_{M^{4,4}(R^m)} &\leq C\|\Delta\tilde{u}\|\tilde{A}\|_{M^{\frac{4}{3},4}(R^m)} \\ &\leq C\|\Delta\tilde{u}\|_{M^{2,4}(R^m)}\|\tilde{A}\|_{M^{4,4}(R^m)} \\ &\leq C\|\nabla^2u\|_{M^{2,4}(B_1)}\|\nabla u\|_{M^{4,4}(B_1)}^2 \\ &\leq C\|\nabla u\|_{M^{4,4}(B_1)}^2 \end{aligned}$$

This implies that (4.16) holds for $i = 4$.

For ω_5^α , since

$$\begin{aligned} |\nabla\omega_5^\alpha|(x) &\leq C\int_{R^m}|\nabla G|(x-y)|\Delta\tilde{u}|\nabla\tilde{A}|(y)dy \\ &= CI_3(|\Delta\tilde{u}|\nabla\tilde{A}|)(x) \end{aligned}$$

Hence, by (4.6), we have that $\nabla\omega_5^\alpha \in M_*^{4,4}(R^m)$ and

$$\begin{aligned} \|\nabla\omega_5^\alpha\|_{M_*^{4,4}(R^m)} &\leq C\|\Delta\tilde{u}\|\nabla\tilde{A}\|_{M^{1,4}(R^m)} \\ &\leq C\|\nabla\tilde{A}\|_{M^{2,4}(R^m)}\|\Delta\tilde{u}\|_{M^{2,4}(R^m)} \\ &\leq C\|\nabla^2u\|_{M^{2,4}(B_1)}\|\nabla u\|_{M^{4,4}(B_1)}^2 \\ &\leq C\|\nabla u\|_{M^{4,4}(B_1)}^2 \end{aligned}$$

This implies (4.16) for $i = 5$.

For ω_6^α , since

$$\begin{aligned} |\nabla\omega_6^\alpha|(x) &\leq C\sum_{\beta}\int_{R^m}|\nabla G|(x-y)|(\Delta(\tilde{u}))^T|\nabla\tilde{e}_\beta|\tilde{A}^{\alpha\beta}|(y)dy \\ &\leq CI_3((|\Delta\tilde{u}|+|\nabla\tilde{u}|^2)(|\tilde{A}|^2+|\nabla\tilde{u}|^2))(x) \end{aligned}$$

Hence (4.6) implies that $\nabla\omega_6^\alpha \in M_*^{4,4}(R^m)$, and, by (4.11) (4.12) and (4.14),

$$\begin{aligned} \|\nabla\omega_6^\alpha\|_{M_*^{4,4}(R^m)} &\leq C\|(|\nabla^2\tilde{u}|+|\nabla\tilde{u}|^2)\|_{M^{2,4}(R^m)}(\|\tilde{A}\|_{M^{2,4}(R^m)}^2+\|\nabla\tilde{u}\|_{M^{2,4}(R^m)}^2) \\ &\leq C\|(|\Delta\tilde{u}|+|\nabla\tilde{u}|^2)\|_{L^2,\infty(R^4)} \\ &\leq C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)}^2 \end{aligned}$$

Similarly, for ω_7^α and ω_8^α , since

$$\begin{aligned} |\nabla\omega_7^\alpha|(x) &\leq C\int_{R^m}|\nabla G(x-y)|(|\Delta\tilde{u}|+|\nabla\tilde{u}|^2)(|\tilde{A}|^2+|\nabla\tilde{u}|^2)(y)dy \\ &\leq CI_3((|\nabla^2\tilde{u}|+|\nabla\tilde{u}|^2)(|\tilde{A}|^2+|\nabla\tilde{u}|^2))(x) \end{aligned}$$

$$\begin{aligned}
|\nabla\omega_8^\alpha|(x) &\leq C \int_{R^m} |\nabla G|(x-y)(|\nabla^2\tilde{u}||\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^3)(|\tilde{A}| + |\nabla\tilde{u}|)(y) \\
&= CI_3((|\nabla^2\tilde{u}||\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^3)(|\tilde{A}| + |\nabla\tilde{u}|))(x)
\end{aligned}$$

we can prove (4.16) holds for $i = 7, 9$ in the same way as that for $i = 6$.

Finally, for ω_9^α , since

$$\begin{aligned}
|\nabla\omega_9^\alpha|(x) &\leq C \int_{R^m} |\nabla^2 G|(x-y)|\nabla\tilde{u}|^2(|\tilde{A}| + |\nabla\tilde{u}|)(y) dy \\
&= CI_2(|\nabla\tilde{u}|^2(|\tilde{A}| + |\nabla\tilde{u}|))(x)
\end{aligned}$$

Hence it follows from (4.4) that $\nabla\omega_9^\alpha \in M^{4,4}(R^m)$ and

$$\begin{aligned}
\|\nabla\omega_9^\alpha\|_{M_*^{4,4}(R^m)} &\leq \|\nabla\omega_9^\alpha\|_{M^{4,4}(R^m)} \\
&\leq C\| |\nabla\tilde{u}|^2(|\tilde{A}| + |\nabla\tilde{u}|)\|_{M^{\frac{4}{3},3}(R^m)} \\
&\leq C\|\nabla\tilde{u}\|_{M^{4,4}(R^m)}^2(\|\tilde{A}\|_{M^{4,4}(R^m)} + \|\nabla\tilde{u}\|_{M^{4,4}(R^m)}) \\
&\leq C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)}^2
\end{aligned}$$

Therefore, we have verified that (4.16) holds for all $1 \leq i \leq 9$.

Now, we consider the Hodge decomposition of the one form $\langle d\tilde{u}, \tilde{e}_\alpha \rangle$. It is well-known (cf. [IM]) that there are a f_α and a two form g_α such that $\nabla f_\alpha, \nabla g_\alpha \in M^{4,4}(R^m)$, and

$$\langle d\tilde{u}, \tilde{e}_\alpha \rangle = df_\alpha + d^*g_\alpha, dg_\alpha = 0, \quad (4.19)$$

$$\|\nabla f_\alpha\|_{M_*^{4,4}(R^m)} + \|\nabla g_\alpha\|_{M_*^{4,4}(R^m)} \leq C\|\nabla\tilde{u}\|_{M_*^{4,4}(R^m)} \quad (4.20)$$

By taking contractions of the eqn. (4.19) with respect to d and d^* , we have

$$\Delta^2 g_\alpha = \Delta(d\tilde{u} \wedge d\tilde{e}_\alpha), \text{ in } R^m, \quad (4.21)$$

$$\Delta^2 f_\alpha = \Delta\nabla \cdot \langle \nabla u, e_\alpha \rangle = \sum_{j=1}^9 \Delta^2 \omega_j^\alpha, \text{ in } B_1 \quad (4.22)$$

Therefore, $f_\alpha - \sum_{j=1}^9 \omega_j^\alpha$ is a biharmonic function in B_1 , i.e.

$$\Delta^2(f_\alpha - \sum_{j=1}^9 \omega_j^\alpha) = 0, \text{ in } B_1 \quad (4.23)$$

Now we need to estimate both $\|\nabla f_\alpha\|_{M_*^{4,4}(R^m)}$ and $\|\nabla g_\alpha\|_{M_*^{4,4}(R^m)}$. Since

$$|\nabla g_\alpha|(x) \leq C \int_{R^m} |\nabla^2 G|(x-y)|\nabla\tilde{u}||\nabla\tilde{e}_\alpha|(y) \leq CI_2(|\nabla\tilde{u}|(|\tilde{A}| + |\nabla\tilde{u}|))(x)$$

we have

$$\begin{aligned}\|\nabla g_\alpha\|_{M_*^{4,4}(R^m)} &\leq C\|\nabla\tilde{u}\|_{M^{4,4}(R^m)}(\|\tilde{A}\|_{M^{4,4}(R^m)} + \|\nabla\tilde{u}\|_{M^{4,4}(R^m)}) \\ &\leq C\|\nabla u\|_{M^{4,4}(B_1)}^2\end{aligned}\quad (4.24)$$

For ∇f_α , we have, by (4.16) and the standard estimate for biharmonic functions, that, for any $\theta \in (0, \frac{1}{2})$,

$$\begin{aligned}\|\nabla f_\alpha\|_{M_*^{4,4}(B_\theta)} &\leq \|\nabla(f_\alpha - \sum_{i=1}^9 \omega_i^\alpha)\|_{M_*^{4,4}(B_\theta)} + \sum_{i=1}^9 \|\nabla\omega_i^\alpha\|_{M_*^{4,4}(B_\theta)} \\ &\leq C\theta\|\nabla(f_\alpha - \sum_{i=1}^9 \omega_i^\alpha)\|_{M_*^{4,4}(B_1)} + \sum_{i=1}^9 \|\nabla\omega_i^\alpha\|_{M_*^{4,4}(R^m)} \\ &\leq C\theta\|\nabla f_\alpha\|_{M_*^{4,4}(R^m)} + C(1+\theta)\sum_{i=1}^9 \|\nabla\omega_i^\alpha\|_{M_*^{4,4}(R^m)} \\ &\leq C\theta\|\nabla f_\alpha\|_{M_*^{4,4}(R^m)} + C\|\nabla u\|_{M^{4,4}(B_1)}^2 \\ &\leq C\theta\|\nabla u\|_{M_*^{4,4}(B_1)} + C\|\nabla u\|_{M^{4,4}(B_1)}^2\end{aligned}\quad (4.25)$$

Therefore, we have

$$\begin{aligned}\|\nabla u\|_{M_*^{4,4}(B_\theta)} &\leq C(\|\nabla f_\alpha\|_{M_*^{4,4}(B_\theta)} + \|\nabla g_\alpha\|_{M_*^{4,4}(B_\theta)}) \\ &\leq C\theta\|\nabla u\|_{M_*^{4,4}(B_1)} + C\|\nabla u\|_{M_*^{4,4}(B_1)}^2\end{aligned}\quad (4.26)$$

To estimate $\|\nabla u\|_{M^{4,4}(B_1)}^2$, we need to apply proposition 4.3. Since $\nabla u \in M^{4,4}(B_1) \subset M^{2,2}(B_1)$ and $\nabla^2 u \in M^{2,4}(B_1)$, (4.7) implies that

$$\|\nabla u\|_{M^{4,4}(B_1)}^2 \leq C[u]\text{BMO}_{(B_1)}(\|\nabla u\|_{M^{2,2}(B_1)} + \|\nabla^2 u\|_{M^{2,4}(B_1)}) \leq C\epsilon_0[u]\text{BMO}_{(B_1)} \quad (4.27)$$

On other hand, the Poincaré inequality implies

$$[u]\text{BMO}_{\bar{B}(B_1)} \leq C\|\nabla u\|_{M_*^{4,4}(B_1)} \quad (4.28)$$

Therefore, combining (4.26), (4.27), and (4.28) together, we have

$$\|\nabla u\|_{M_*^{4,4}(B_\theta)} \leq C(\theta + \epsilon_0)\|\nabla u\|_{M_*^{4,4}(B_1)} \quad (4.29)$$

this clearly implies (4.2) provided that both $\theta_0 = \theta \in (0, \frac{1}{2})$ and $\epsilon_0 > 0$ are chosen to be sufficiently small. Therefore, the proof of Lemma 4.1 is complete. \blacksquare

We end this section by indicating how to obtain the Hölder continuity of biharmonic maps which satisfy the smallness condition (4.1) of Lemma 4.1.

Proposition 4.6. *There exists an $\epsilon_0 > 0$ such that if $u \in W^{2,2}(B_2, N)$ is an extrinsic (or intrinsic, respectively) biharmonic map such that u^*TN is trivial and*

$$\|\nabla u\|_{M^{4,4}(B_2)} + \|\nabla^2 u\|_{M^{2,4}(B_2)} \leq \epsilon_0 \quad (4.30)$$

then $u \in C^\infty(B_1, N)$

Proof. First, since (4.30) implies

$$\sup_{x \in B_1, 0 < r \leq 1} (\|\nabla u\|_{M^{4,4}(B_r(x))} + \|\nabla^2 u\|_{M^{2,4}(B_r(x))}) \leq \epsilon_0$$

therefore, we can repeatedly apply Lemma 4.1 to conclude that there is an $\theta_0 \in (0, \frac{1}{2})$ such that, for any $x \in B_1$ and $l \geq 1$,

$$\|\nabla u\|_{M_*^{4,4}(B_{\theta_0^l}(x))} \leq 2^{-l} \|\nabla u\|_{M_*^{4,4}(B_1(x))} \leq 2^{-l} \|\nabla u\|_{M_*^{4,4}(B_2)}$$

Therefore there exists an $\alpha_0 \in (0, 1)$ such that

$$\|\nabla u\|_{M_*^{4,4}(B_r(x))} \leq Cr^{\alpha_0}, \quad \forall x \in B_1, 0 < r \leq \frac{1}{2}$$

This, combined with Morrey Lemma, implies that $u \in C^{\alpha_0}(B_1, N)$. Now, we can apply the higher order regularity theorem 5.1 of [CWY] to conclude that $u \in C^\infty(B_1, N)$. \blacksquare

§5. Monotonicity inequality and proof of theorem A

This section is devoted to deviation of a monotonicity inequality for stationary biharmonic maps. Then, we establish an interior estimate of weakly biharmonic maps to control boundary terms in the monotonicity inequality. Finally, we prove theorem A by showing that, away from a closed set Σ of zero H^{m-4} -measure, $\|\nabla u\|_{M^{4,4}(B_r(x))} + \|\nabla^2 u\|_{M^{2,4}(B_r(x))}$ tends to zero as r shrinks to zero.

Since such a monotonicity inequality was already established by [CWY](proposition 3.2) for stationary extrinsic biharmonic maps, we will focus on the deviation for stationary intrinsic biharmonic maps. Note that we don't need to assume the triviality of u^*TN in the following Lemma.

Lemma 5.1. *If $u \in W^{2,2}(\Omega, N)$ is a stationary intrinsic (or extrinsic, respectively) biharmonic map. Then, for any $X \in C_0^1(\Omega, R^m)$, we have*

$$\begin{aligned} & \int_{\Omega} X(|\Delta u|^2) + (\nabla \cdot X)|\Delta u|^2 \\ &= \int_{\Omega} 4B(u)(\nabla u, \nabla u) \sum_{ij=1}^m B(u)(u_i, u_j) X_j^i - |B(u)(\nabla u, \nabla u)|^2 \nabla \cdot X \end{aligned} \quad (5.1)$$

Proof. One can refer to [CWY] proposition 3.1 for the proof of (5.1) when u is an extrinsic biharmonic maps. Here we assume that u is a stationary intrinsic biharmonic map. For $X \in C_0^1(\Omega, R^m)$, denote $u_t(x) = u(x + tX(x))$. First, observe that

$$|(\Delta u_t)^T|^2 = |\Delta u_t|^2 - |B(u_t)(\nabla u_t, \nabla u_t)|^2$$

Therefore, the stationarity of u implies

$$\frac{d}{dt}\Big|_{t=0} \int_{\Omega} |\Delta u_t|^2 = \frac{d}{dt}\Big|_{t=0} \int_{\Omega} |B(u_t)(\nabla u_t, \nabla u_t)|^2$$

As in Lemma 3.1 of [CWY], we have

$$\frac{d}{dt}\Big|_{t=0} \int_{\Omega} |\Delta u_t|^2 = \int_{\Omega} X(|\Delta u|^2) + (\nabla \cdot X)|\Delta u|^2$$

For the right hand side of (5.2), we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_{\Omega} |B(u_t)(\nabla u_t, \nabla u_t)|^2 &= \int_{\Omega} \left(4 \sum_{i,j=1}^m B(u)(\nabla u, \nabla u) B(u)(u_i, u_j) X_j^i \right. \\ &\quad \left. - |B(u)(\nabla u, \nabla u)|^2 \nabla \cdot X \right) \end{aligned}$$

Putting these two identities together, we prove (5.1). ■

Lemma 5.2. *Suppose that $u \in W^{2,2}(\Omega, N)$ is a stationary extrinsic (or intrinsic, respectively) biharmonic map. Then, for any $x \in \Omega$ and $0 < r < R < \text{dist}(x, \partial\Omega)$, it holds*

$$\begin{aligned} &R^{4-m} \int_{B_R(x)} |\Delta u|^2 - r^{4-m} \int_{B_r(x)} |\Delta u|^2 \\ &\geq 2 \int_{\partial B_R(x)} \left(-\frac{x_i u_l u_{il}}{|x|^{m-2}} + 2 \frac{(x_i u_i)^2}{|x|^{m-1}} - 2 \frac{|\nabla u|^2}{|x|^{m-3}} \right) dH^{m-1} \\ &\quad - 2 \int_{\partial B_r(x)} \left(-\frac{x_i u_l u_{il}}{|x|^{m-2}} + 2 \frac{(x_i u_i)^2}{|x|^{m-1}} - 2 \frac{|\nabla u|^2}{|x|^{m-3}} \right) dH^{m-1} \end{aligned} \quad (5.2)$$

Proof. Since (5.2) was proved by [CWY] proposition 3.2 for extrinsic biharmonic maps, we assume that u is a stationary intrinsic biharmonic map. For simplicity, let $x = 0 \in \Omega$ and $X = \sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$. First, observe that by choosing suitable cut-off functions, (5.1) implies that for a.e. $r > 0$,

$$\begin{aligned} &\int_{B_r} X(|\Delta u|^2) + m|\Delta u|^2 - r \int_{\partial B_r} |\Delta u|^2 \\ &= (4-m) \int_{B_r} |B(u)(\nabla u, \nabla u)|^2 + r \int_{\partial B_r} |B(u)(\nabla u, \nabla u)|^2 \\ &\quad - 4r \int_{\partial B_r} B(u)(\nabla u, \nabla u) B(u) \left(\frac{\partial u}{\partial r}, \frac{\partial u}{\partial r} \right) \end{aligned} \quad (5.3)$$

It follows from (5.3) that

$$\begin{aligned}
& r^{m-3} \frac{d}{dr} (r^{4-m} \int_{B_r} |\Delta u|^2) \\
&= r \int_{\partial B_r} |\Delta u|^2 + (4-m) \int_{B_r} |\Delta u|^2 \\
&= \int_{B_r} (X(|\Delta u|^2) + 4|\Delta u|^2) + (m-4) \int_{B_r} |B(u)(\nabla u, \nabla u)|^2 \\
&\quad - r \int_{\partial B_r} |B(u)(\nabla u, \nabla u)|^2 + 4r \int_{\partial B_r} B(u)(\nabla u, \nabla u) B(u) \left(\frac{\partial u}{\partial r}, \frac{\partial u}{\partial r} \right) \tag{5.4}
\end{aligned}$$

Note that the eqn. (2.3), for intrinsic biharmonic maps, yields

$$\begin{aligned}
x_i u_i \Delta^2 u &= x_i u_i B(u)(\nabla u, \nabla u) \nabla_u B(u)(\nabla u, \nabla u) \\
&\quad + 2B(u)(\nabla u, \nabla u) B(u)(\nabla u, \nabla(x_i u_i)) \\
&\quad - 2\nabla \cdot (B(u)(\nabla u, \nabla u) B(u)(\nabla u, x_i u_i)) \\
&= 2|B(u)(\nabla u, \nabla u)|^2 + x_i B(u)(\nabla u, \nabla u) \frac{\partial}{\partial x_i} (B(u)(\nabla u, \nabla u)) \\
&\quad - 2\nabla \cdot (B(u)(\nabla u, \nabla u) B(u)(\nabla u, x_i u_i)) \tag{5.5}
\end{aligned}$$

By using (5.5), we now proceed in the same way as that of proposition 3.2 of [CWY] to estimate

$$\begin{aligned}
\int_{B_r} (X(|\Delta u|^2) + 4|\Delta u|^2) &= \int_{B_r} (2x_i (\Delta u)_i (\Delta u) + 4|\Delta u|^2) \\
&= \int_{\partial B_r} \frac{2x_i x_k u_{ik} \Delta u - 2x_i x_k u_i (\Delta u)_k}{r} \\
&\quad + \int_{B_r} (2|\Delta u|^2 + 2(\Delta u)_k u_k + 2x_i u_i \Delta^2 u) \\
&= \int_{\partial B_r} \frac{2x_i x_k u_{ik} \Delta u - 2x_i x_k u_i (\Delta u)_k + 2x_k u_k \Delta u}{r} \\
&\quad + \int_{B_r} 4|B(u)(\nabla u, \nabla u)|^2 + x_i \frac{\partial}{\partial x_i} (|B(u)(\nabla u, \nabla u)|^2) \\
&\quad - 4r \int_{\partial B_r} B(u)(\nabla u, \nabla u) B(u) \left(\frac{\partial u}{\partial r}, \frac{\partial u}{\partial r} \right) \\
&= \int_{\partial B_r} \frac{2x_i x_k u_{ik} \Delta u + 2x_k u_k \Delta u}{r} \\
&\quad + (4-m) \int_{B_r} |B(u)(\nabla u, \nabla u)|^2 + r \int_{\partial B_r} |B(u)(\nabla u, \nabla u)|^2 \\
&\quad - 4r \int_{\partial B_r} B(u)(\nabla u, \nabla u) B(u) \left(\frac{\partial u}{\partial r}, \frac{\partial u}{\partial r} \right) \tag{5.6}
\end{aligned}$$

Inserting (5.6) into (5.4), we obtain

$$r^{m-3} \frac{d}{dr} (r^{4-m} \int_{B_r} |\Delta u|^2) = \int_{\partial B_r} \frac{2x_i x_k u_{ik} \Delta u - 2x_i x_k u_i (\Delta u)_k + 2x_k u_k \Delta u}{r}$$

Now we can follow exactly the arguments (3.3)–(3.8) of [CWY] to get (5.2). ■

Now we want to apply the inequality (5.2) of Lemma 5.2 to prove that the condition (1.1) holds for H^{m-4} a.e. $x \in \Omega$. More precisely, we need to have

Lemma 5.3. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, 1)$ such that if $u \in W^{2,2}(\Omega, N)$ is a stationary extrinsic (or intrinsic, respectively) biharmonic map satisfying, for $B_r \subset \Omega$,*

$$r^{4-m} \int_{B_r} |\nabla^2 u|^2 + r^{2-m} \int_{B_r} |\nabla u|^2 \leq \epsilon_0^2 \quad (5.7)$$

then

$$\|\nabla u\|_{M^{4,4}(B_{\theta_0 r})} + \|\nabla^2 u\|_{M^{2,4}(B_{\theta_0 r})} \leq C\epsilon_0 \quad (5.8)$$

Proof. We first observe that, by an interpolation inequality of Nirenberg [N], we have, for all $2 \leq q \leq 4$,

$$\begin{aligned} (r^{q-m} \int_{B_r} |\nabla u|^q)^{\frac{1}{q}} &\leq (r^{4-m} \int_{B_r} |\nabla u|^4)^{\frac{1}{4}} \\ &\leq C \|u\|_{L^\infty(B_r)}^{\frac{1}{2}} (r^{4-m} \int_{B_r} |\nabla^2 u|^2 + r^{2-m} \int_{B_r} |\nabla u|^2)^{\frac{1}{2}} \\ &\leq C\epsilon_0 \end{aligned} \quad (5.9)$$

It is clear that (5.2), combined with the Bochner identity:

$$\Delta |\nabla u|^2 = 2(|\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle)$$

implies that, for any $x \in \Omega$ and $0 < r < R < \text{dist}(x, \partial\Omega)$, we have

$$\begin{aligned} r^{4-m} \int_{B_r(x)} |\nabla^2 u|^2 &\leq R^{4-m} \int_{B_R(x)} |\nabla^2 u|^2 \\ &\quad + C(1 + R^{5-m} \int_{\partial B_R(x)} |\nabla^2 u|^2) R^{3-m} \int_{\partial B_R(x)} |\nabla u|^2 \\ &\quad + C(1 + r^{5-m} \int_{\partial B_r(x)} |\nabla^2 u|^2) r^{3-m} \int_{\partial B_r(x)} |\nabla u|^2 \end{aligned} \quad (5.10)$$

It follows from the Fubini's theorem that for good slices r, R (cf. (4.4) of [CWY]) we have, for $s = r, R$,

$$\begin{aligned} s^{5-m} \int_{\partial B_s(x)} |\nabla^2 u|^2 &\leq 8s^{4-m} \int_{B_{2s}(x)} |\nabla^2 u|^2 \\ s^{3-m} \int_{\partial B_s(x)} |\nabla u|^2 &\leq 8s^{2-m} \int_{B_{2s}(x)} |\nabla u|^2 \end{aligned}$$

Therefore, (5.10) can imply (5.8) by following the induction argument as in Lemma 4.8 of [CWY], if we can control $r^{4-m} \int_{B_{2r}(x)} |\nabla u|^2$ suitably. More precisely, we need the following lemma, a weak type interior estimate for biharmonic maps, to replace Lemma 4.8 of [CWY] during the induction argument.

Lemma 5.4. *suppose that $u \in W^{2,2}(\Omega, N)$ is an extrinsic (or intrinsic, respectively) biharmonic map. Then, for any $0 < \theta < 1$ and $B_r \subset \Omega$, we have, for any $1 \leq p < \frac{m}{m-3}$,*

$$\begin{aligned} (\theta r)^{1-\frac{m}{p}} \|\nabla u\|_{L^p(B_{\theta r})} &\leq C\theta r^{1-\frac{m}{4}} \|\nabla u\|_{L^4(B_r)} \\ &+ C\theta^{4-m} [r^{4-m} \|\nabla^2 u\|_{L^2(B_r)}^2 + (1 + r^{2-\frac{m}{2}} \|\nabla u\|_{L^4(B_r)}^2) r^{2-\frac{m}{2}} \|\nabla u\|_{L^4(B_r)}^2] \end{aligned} \quad (5.11)$$

Proof. Since the inequality (5.11) is invariant under scalings, it suffices to prove it for $r = 1$. The proof is similar to that of Lemma 4.1. For simplicity, we only consider the case of intrinsic biharmonic maps. It follows from the proposition 2.2 that u satisfies

$$\begin{aligned} \Delta^2 u &= \Delta(B(u)(\nabla u, \nabla u)) + 2\nabla \cdot (\langle \nabla(P(u)), \Delta u \rangle) - \langle \Delta(P(u)), \Delta u \rangle \\ &+ P(u)[B(u)(\nabla u, \nabla u)\nabla_u B(\nabla u, \nabla u)] + 2B(u)(\nabla u, \nabla u)B(u)(\nabla u, \nabla(P(u))) \end{aligned}$$

Let $\tilde{u} \in W^{2,2}(R^m)$ be an extension of u such that

$$\|\nabla \tilde{u}\|_{L^4(R^m)} \leq C\|\nabla u\|_{L^4(B_1)}, \quad \|\nabla^2 \tilde{u}\|_{L^2(R^m)} \leq C\|\nabla^2 u\|_{L^2(B_1)} \quad (5.12)$$

and \tilde{B}, \tilde{P} be smooth extensions of B, P from N to R^k . Now we define five auxiliary functions ω_i for $1 \leq i \leq 5$ as follows.

$$\begin{aligned} \omega_1(x) &= \int_{R^m} G(x-y) \Delta(\tilde{B}(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u}))(y) dy \\ \omega_2(x) &= 2 \int_{R^m} G(x-y) \nabla \cdot (\langle \nabla(\tilde{P}(\tilde{u})), \Delta \tilde{u} \rangle)(y) dy \\ \omega_3(x) &= - \int_{R^m} G(x-y) \langle \Delta(\tilde{P}(\tilde{u})), \Delta \tilde{u} \rangle(y) dy \\ \omega_4(x) &= \int_{R^m} G(x-y) \tilde{P}(\tilde{u}) [\tilde{B}(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u})\nabla_{\tilde{u}} \tilde{B}(\nabla \tilde{u}, \nabla \tilde{u})](y) dy \\ \omega_5(x) &= 2 \int_{R^m} G(x-y) \tilde{B}(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u}) \tilde{B}(\tilde{u})(\nabla \tilde{u}, \nabla(\tilde{P}(\tilde{u}))) (y) dy \end{aligned}$$

Then, we have

$$\begin{aligned}
|\nabla\omega_1|(x) &\leq CI_1(|\nabla\tilde{u}|^2)(x) \\
|\nabla\omega_2|(x) &\leq CI_2(|\nabla\tilde{u}||\nabla^2\tilde{u}|)(x) \\
|\nabla\omega_3|(x) &\leq CI_3(|\nabla^2\tilde{u}| + |\nabla\tilde{u}|^4)(x) \\
|\nabla\omega_4|(x) &\leq CI_3(|\nabla\tilde{u}|^4)(x) \\
|\nabla\omega_5|(x) &\leq CI_3(|\nabla\tilde{u}|^4)(x)
\end{aligned}$$

Now we apply (4.6) to estimate $\omega_3, \omega_4, \omega_5$, with $\lambda = m$, to conclude that $\nabla\omega_i \in L_*^{\frac{m}{m-3}}(R^m)$ for $i = 3, 4, 5$ and

$$\begin{aligned}
\sum_{i=3}^5 \|\nabla\omega_i\|_{L_*^{\frac{m}{m-3}}(R^m)} &\leq C \int_{R^m} (|\nabla^2\tilde{u}|^2 + |\nabla\tilde{u}|^4) \\
&\leq C \int_{B_1} (|\nabla^2u|^2 + |\nabla u|^4)
\end{aligned} \tag{5.13}$$

For ω_1, ω_2 , we apply (4.4), with $\lambda = m$, to conclude that $\nabla\omega_1 \in L^{\frac{2m}{m-2}}(R^m)$, $\nabla\omega_2 \in L^{\frac{4m}{3m-8}}(R^m)$, and

$$\begin{aligned}
\|\nabla\omega_1\|_{L^{\frac{2m}{m-2}}(R^m)} &\leq C \|\|\nabla\tilde{u}\|\|_{L^2(R^m)} \\
&\leq C \|\|\nabla\tilde{u}\|\|_{L^4(R^m)}^2 \\
&\leq C \|\|\nabla u\|\|_{L^4(B_1)}^2
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
\|\nabla\omega_2\|_{L^{\frac{4m}{3m-8}}(R^m)} &\leq C \|\|\nabla\tilde{u}\|\|\nabla^2\tilde{u}\|\|_{L^{\frac{4}{3}}(R^m)} \\
&\leq C \|\|\nabla\tilde{u}\|\|_{L^4(R^m)} \|\|\nabla^2\tilde{u}\|\|_{L^2(R^m)} \\
&\leq C (\|\|\nabla u\|\|_{L^4(B_1)}^2 + \|\|\nabla^2u\|\|_{L^2(B_1)}^2)
\end{aligned} \tag{5.15}$$

Therefore, we have

$$\begin{aligned}
\|\nabla\omega_1\|_{L_*^{\frac{m}{m-3}}(B_1)} &\leq C \|\|\nabla\omega_1\|\|_{L^{\frac{2m}{m-2}}(B_1)} \\
&\leq C \|\|\nabla\omega_1\|\|_{L^{\frac{2m}{m-2}}(R^m)} \\
&\leq C \|\|\nabla u\|\|_{L^4(B_1)}^2
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
\|\nabla\omega_2\|_{L_*^{\frac{m}{m-3}}(B_1)} &\leq C \|\|\nabla\omega_1\|\|_{L^{\frac{4m}{3m-8}}(B_1)} \\
&\leq C \|\|\nabla\omega_1\|\|_{L^{\frac{4m}{3m-8}}(R^m)} \\
&\leq C (\|\|\nabla u\|\|_{L^4(B_1)}^2 + \|\|\nabla^2u\|\|_{L^2(B_1)}^2)
\end{aligned} \tag{5.17}$$

Putting these estimates together, we have

$$\sum_{i=1}^5 \|\nabla \omega_i\|_{L_*^{\frac{m}{m-3}}(B_1)} \leq C((1 + \|\nabla u\|_{L^4(B_1)}^2) \|\nabla u\|_{L^4(B_1)}^2 + \|\nabla^2 u\|_{L^2(B_1)}^2) \quad (5.18)$$

It follows from the definition of ω_i 's that

$$\Delta^2(u - \sum_{i=1}^5 \omega_i) = 0, \text{ in } B_1 \quad (5.19)$$

so that the standard estimate for biharmonic functions implies that for any $\theta \in (0, 1)$

$$\begin{aligned} \theta^{4-m} \|\nabla(u - \sum_{i=1}^5 \omega_i)\|_{L_*^{\frac{m}{m-3}}(B_\theta)} &\leq C\theta \|\nabla(u - \sum_{i=1}^5 \omega_i)\|_{L_*^{\frac{m}{m-3}}(B_1)} \\ &\leq C\theta \|\nabla u\|_{L_*^{\frac{m}{m-3}}(B_1)} + C \sum_{i=1}^5 \|\nabla \omega_i\|_{L_*^{\frac{m}{m-3}}(B_1)} \end{aligned}$$

This implies

$$\begin{aligned} \theta^{4-m} \|\nabla u\|_{L_*^{\frac{m}{m-3}}(B_\theta)} &\leq C\theta \|\nabla u\|_{L_*^{\frac{m}{m-3}}(B_1)} + C\theta^{4-m} \sum_{i=1}^5 \|\nabla \omega_i\|_{L_*^{\frac{m}{m-3}}(B_1)} \\ &\leq C\theta \|\nabla u\|_{L^4(B_1)} + C\theta^{4-m} ((1 + \|\nabla u\|_{L^4(B_1)}^2) \|\nabla u\|_{L^4(B_1)}^2 + \|\nabla^2 u\|_{L^2(B_1)}^2) \end{aligned}$$

where we have used the inequality $\|\nabla u\|_{L_*^{\frac{m}{m-3}}(B_\theta)} \leq C\|\nabla u\|_{L^4(B_1)}$ in the last step. This, combined with

$$\theta^{1-\frac{m}{p}} \|\nabla u\|_{L^p(B_\theta)} \leq \theta^{4-m} \|\nabla u\|_{L_*^{\frac{m}{m-3}}(B_\theta)}, \quad \forall 1 < p < \frac{m}{m-3}$$

implies (5.11). Therefore, the proof of Lemma 5.4 is complete. \blacksquare

Completion of the proof of theorem A.

Define

$$\Sigma = \{x \in \Omega : \limsup_{r \downarrow 0} r^{2-m} (\int_{B(x)} r^2 |\nabla^2 u|^2 + |\nabla u|^2) \geq \epsilon_0^2\}$$

where $\epsilon_0 > 0$ is the same number as in (4.6). We first observe that (4.8) implies $\nabla u \in L^4(\Omega)$ so that, we have, by the Hölder inequality, $\Sigma \subset \Sigma_1$, where

$$\Sigma_1 = \{x \in \Omega : \limsup_{r \downarrow 0} r^{4-m} (\int_{B(x)} |\nabla^2 u|^2 + |\nabla u|^4) \geq \epsilon_0^2\}$$

By Federer-Ziemmer (cf. [Z]), we have $H^{m-4}(\Sigma_1) = 0$ so that $H^{m-4}(\Sigma) = 0$ as well. Now, for any $x_0 \in \Omega \setminus \Sigma$, there exists a $r_0 > 0$ such that

$$r_0^{2-m} \int_{B_{r_0}(x_0)} r_0^2 |\nabla^2 u|^2 + |\nabla u|^2 < \epsilon_0^2$$

Hence Lemma 5.3 implies that there exists a $\delta_0 \in (0, 1)$ such that, for any $y \in B_{\frac{r_0}{4}}(x_0)$,

$$\|\nabla u\|_{M^{4,4}(B_{\delta_0 r_0}(y))} + \|\nabla^2 u\|_{M^{2,4}(B_{\delta_0 r_0}(y))} \leq C\epsilon_0$$

Therefore, we can apply proposition 4.6, with u replaced by $u_{x,r_0}(z) \equiv u(x + r_0 z)$, to conclude that $u \in C^\infty(B_{\frac{r_0}{4}}(x_0), N)$. In particular, $u \in C^\infty(\Omega \setminus \Sigma, N)$. This finishes the proof of theorem A. ■

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