Remarks on biharmonic maps into spheres

Changyou Wang
Department of Mathematics, University of Kentucky
Lexington, KY 40506, USA

Abstract. *We prove an apriori estimate in Morrey spaces for both intrinsic and extrinsic biharmonic maps into spheres. As applications, we prove an energy quantization theorem for biharmonic maps from 4-manifolds into spheres and a partial regularity for stationary intrinsic biharmonic maps into spheres.*

§1. Introduction

Let $\Omega \subset R^m$ be a bounded domain and $S^{k-1} \subset R^k$ be the standard unit sphere. $W^{2,2}(\Omega, S^{k-1})$ is defined by

$$W^{2,2}(\Omega, S^{k-1}) = \{v \in W^{2,2}(\Omega, R^k) : v(x) \in S^{k-1} \text{ for a.e. } x \in \Omega\}$$

There are two important second-order energy functionals on $W^{2,2}(\Omega, S^{k-1})$, namely, the hessian energy functional and the tension field energy functional. More precisely,

$$H(u) = \int_\Omega |\Delta u|^2 \, dx, \quad T(u) = \int_\Omega |(\Delta u)^T|^2 \, dx$$

where $(\Delta u)^T$ is the component of Δu which is tangent to S^{k-1} at u. Note that $(\Delta u)^T$ is also called the tension field of u in the literature (cf. [EL]). Motivated by the analogy of harmonic maps and potential applications to higher dimensional conformal geometry, Chang-Wang-Yang, in [CWY], began an analytic study of the regularity properties of extrinsic biharmonic maps (i.e. critical points of $H(\cdot)$) in dimension great than or equal to four, and established a regularity theorem in dimension four and a partial regularity theorem for stationary extrinsic biharmonic maps in dimensions greater than four. Later, Ku, in [K], proved regularity for intrinsic biharmonic maps (i.e. critical points of $T(\cdot)$) in dimension four. The readers may compare [CWY] with the corresponding theorem by Evans [E] (cf. also [CWY1]) on stationary harmonic maps into spheres. This naturally leads us to investigate regularity issues of biharmonic maps into general Riemannian manifolds N, because of their similarities to the theorems by Hélein [H] and Bethuel [B] on stationary harmonic maps. In two forthcoming articles [W1] [W2], the author shows, in [W1], that any extrinsic (or intrinsic, respectively) biharmonic map $u \in W^{2,2}(\Omega, N)$ is smooth in dimension 4, provided that u^*TN is trivial. For $m \geq 5$, we also prove, in [W2], a partial regularity theorem for stationary biharmonic maps, which asserts that if $u \in W^{2,2}(\Omega, N)$ is a stationary extrinsic (or intrinsic, respectively) biharmonic maps u such that u^*TN is trivial then u is smooth away from a closed set of $(m-4)$-dimensional Hausdorff measure zero. Although we need the Coulomb gauge construction techniques (due to Uhlenbeck [U1] [U1] for $m=4$ and Meyer-Rivieré [MR], Tao-Tian [TT] for $m \geq 5$) in [W1] [W2], the another key ingredient is the decay Lemma in Morrey spaces under the smallness of the hessian energy, that we demonstrate here in detail for $N = S^{k-1}$.

In order to state the apriori estimate for biharmonic maps, we first recall the definition of Morrey spaces (cf. [A]).

Definition 1. For an open set $E \subset R^m$, $1 \leq p < \infty$, and $0 < \lambda \leq m$, the Morrey space $M^{p,\lambda}(E)$ is defined by

$$M^{p,\lambda}(E) = \{f \in L^p(E) : \|f\|^p_{M^{p,\lambda}(E)} \equiv \sup_{B_r \subset E} \{r^{\lambda - m} \int_{B_r} |f|^p(y)\, dy\} < \infty\}$$

It is clear that $M^{p,m}(E) = L^p(E)$.

Definition 2. (a) A map $u \in W^{2,2}(\Omega, S^{k-1})$ is called an extrinsic biharmonic map if u is a critical point of the hessian energy $H(\cdot)$; (b) A map $u \in W^{2,2}(\Omega, S^{k-1})$ is called an intrinsic biharmonic map if u is a critical point of the tension field energy $T(\cdot)$.

Now we state our main theorem.

Theorem A. *There exist $\epsilon_0 > 0$, $\theta_0 \in (0, \frac{1}{2})$ such that if $u \in W^{2,2}(\Omega, S^{k-1})$ is an extrinsic (or intrinsic, respectively) biharmonic map and satisfies, for $B_r(x) \subset \Omega$,*

$$\|\nabla u\|^4_{M^{4,4}(B_r(x))} + \|\nabla^2 u\|^2_{M^{2,4}(B_r(x))} \leq \epsilon_0^2 \tag{1.1}$$

then

$$\|\nabla u\|_{M^{4,4}(B_{\theta_0 r}(x))} \leq \frac{1}{2} \|\nabla u\|_{M^{4,4}(B_r(x))} \tag{1.2}$$

In particular, $u \in C^\infty(B_{\frac{r}{2}}(x), S^{k-1})$.

Note that (1.2), combined with iterations and the Morrey Lemma, yields that $u \in C^\alpha(B_{\frac{r}{2}}(x), S^{k-1})$, for some $\alpha \in (0,1)$. This, combined with the higher order regularity theorem 5.1 of [CWY], yields that $u \in C^\infty(B_{\frac{r}{2}}(x), S^{k-1})$.

Observe that, for $m = 4$,

$$\|\nabla u\|^4_{M^{4,4}(B_r(x))} + \|\nabla^2 u\|^2_{M^{2,4}(B_r(x))} = \int_{B_r(x)} |\nabla u|^4 + |\nabla^2 u|^2$$

this, combined with the absolute continuity of $\int |\nabla u|^4 + |\nabla^2 u|^2$, implies that the condition (1.1) is automatically true for sufficiently small $r_0 = r_0(u, \epsilon_0) > 0$. Therefore, theorem A gives an alternative proof for the smoothness of both extrinsic and intrinsic biharmonic maps from four dimensional domains into spheres, which was originally proved by Chang-Wang-Yang [CWY] and Ku [K] through a different method. As a new application of theorem A, we consider the quantization issue for weakly convergent sequences of biharmonic maps from four dimensional manifolds into spheres. We are able to prove an energy quantization theorem, similar to that of harmonic maps from surfaces (see Jost [J], Parker [P], Qing [Q], Ding-Tian [DT], Lin-Wang [LW], Wang [W], etc).

In order to state the theorem, we first need

Definition 3. A nonconstant map $\omega \in C^\infty(S^4, S^{k-1})$ is called an (i) extrinsic quasi-biharmonic S^4 if it is a critical point of

$$\bar{H}(v) = \int_{S^4} (|\Delta_{S^4} v|^2 + 2|\nabla_{S^4} v|^2)\, dH^4$$

(ii) intrinsic quasi-biharmonic S^4 if it is a critical point of

$$\bar{T}(v) = \int_{S^4} (|(\Delta_{S^4} v)^T|^2 + 2|\nabla_{S^4} v|^2)\, dH^4$$

over $W^{2,2}(S^4, S^{k-1})$.

Let $\Pi : S^4 \to R^4$ denote the stereographic projection from the north pole $(0', 1)$. It is well-known (see, e.g. Chang-Yang [CY]) that, for any $v \in W^{2,2}(R^4, S^{k-1})$,

$$\int_{R^4} |\Delta v|^2(x)\, dx = \int_{S^4} (|\Delta_{S^4} \bar{v}|^2 + 2|\nabla_{S^4} \bar{v}|^2)\, dH^4$$

where $\bar{v} = v \circ \Pi$. Therefore, with the help of Lemma 3.4 below, we have that $\omega \in C^\infty(S^4, S^{k-1})$ is an extrinsic (or intrinsic, respectively) quasi-biharmonic S^4 if and only if $\bar{\omega} \equiv \omega \circ \Pi^{-1} \in C^\infty \cap W^{2,2}(R^4, S^{k-1})$ is a nonconstant extrinsic (or intrinsic, respectively) biharmonic map.

Theorem B. *Let (M, g) be a closed Riemannian manifold of dimension four. Suppose that $\{u_n\} \subset C^\infty(M, S^{k-1})$ is a sequence of extrinsic (or intrinsic, respectively) biharmonic maps converging to $u \in W^{2,2}(M, S^{k-1})$ weakly in $W^{2,2}$. Then $u \in C^\infty(M, S^{k-1})$ is an extrinsic (or intrinsic, respectively) biharmonic map, and there are a set $\Sigma \subset M$ of finite points, an integer $0 \le d \le C_0 < \infty$, and extrinsic (or intrinsic, respectively) quasi-biharmonic S^4, $\{\omega_i\}_{i=1}^d$, such that, after taking possible subsequences, $u_n \to u$ in $W^{2,2}_{loc} \cap C^4_{loc}(M \setminus \Sigma, S^{k-1})$. Moreover,*

$$\lim_{n\to\infty} \int_M |\nabla^2 u_n|_g^2\, dv_g = \int_M |\nabla^2 u|_g^2\, dv_g + \sum_{i=1}^d \int_{R^4} |\nabla^2 \bar{\omega}_i|^2, \tag{1.3}$$

and

$$\lim_{n\to\infty}\int_M |\nabla u_n|_g^4\,dv_g = \int_M |\nabla u|_g^4\,dv_g + \sum_{i=1}^d \int_{R^4} |\nabla\bar{\omega}_i|^4 \tag{1.4}$$

where $\bar{\omega}_i = \omega_i\circ\Pi^{-1}$, $1\le i\le d$.

When $m\ge 5$, it is easy to find biharmonic maps having singularities. For example, one can show that $\frac{x}{|x|}: B^m\to S^{m-1}$ has the properties: (i) the tension energy $T(\frac{x}{|x|})=0$ since $\frac{x}{|x|}$ is a harmonic map; (ii) the hessian energy $H(\frac{x}{|x|}) = \int_{B^m}|\nabla\frac{x}{|x|}|^4$ and $\frac{x}{|x|}$ is a minimizing extrinsic biharmonic map, since $\frac{x}{|x|}$ is a minimizing 4-harmonic map (see, e.g. [CG]). Moreover, motivated by the partial regularity theory for harmonic maps (see, e.g. [SU], [H], [E], [B] [CWY1]), we need to restrict to the class of so-called stationary biharmonic maps. The notation of stationary extrinsic biharmonic maps was introduced by [CWY]. Now we recall it and also define the concept of stationary intrinsic biharmonic maps.

Definition 4. (a) A $W^{2,2}(\Omega,S^{k-1})$-extrinsic biharmonic map u is called stationary if, in addition, u is a critical point of $H(\cdot)$ with respect to the domain variations, i.e. for any $X\in C_0^1(\Omega,R^m)$

$$\frac{d}{dt}|_{t=0}\int_\Omega |\Delta u_t|^2\,dx = 0, \tag{1.5}$$

(b) A $W^{2,2}(\Omega,S^{k-1})$-intrinsic biharmonic map u is called stationary if, in addition, u is a critical point of $T(\cdot)$ with respect to the domain variations, i.e. for any $X\in C_0^1(\Omega,R^m)$

$$\frac{d}{dt}|_{t=0}\int_\Omega |(\Delta u_t)^T|^2\,dx = 0, \tag{1.6}$$

where $u_t(x)=u(x+tX(x))$ for $x\in\Omega$.

The partial regularity for stationary extrinsic biharmonic maps into spheres was obtained by [CWY]. Here we show that theorem A, combined with Lemma 4.3 below, also implies that the same partial regularity theorem for stationary intrinsic biharmonic maps into spheres. Roughly speaking, Lemma 4.3 plays the same role as Lemma 4.8 of [CWY] in the process to control the boundary terms appearing in the monotonicity inequality (4.3) for stationary intrinsic biharmonic maps so that *for* ϵ_0 *and* $\rho_0>0$ *sufficiently small, there exists* $C_0>0$ *such that if*

$$r^{4-m}\int_{B_r(x)} |\nabla u|^4 + |\nabla^2 u|^2 \le \epsilon_0^2 \tag{1.7}$$

then

$$\|\nabla u\|^4_{M^{4,4}(B_{\rho_0 r}(x))} + \|\nabla^2 u\|^2_{M^{2,4}(B_{\rho_0 r}(x))} \le C_0\epsilon_0^2 \tag{1.8}$$

Therefore, theorem A and the standard Vitali's covering argument give

Theorem C. *A stationary intrinsic biharmonic map* $u \in W^{2,2}(\Omega, S^{k-1})$ *is smooth away from a closed set of* $(m-4)$*-dimensional Hausdorff measure zero.*

We point out that theorem A and Lemma 4.3 below can be made to give a different proof of theorem 4.1 of [CWY] as well.

The paper is organized as follows. In §2, we prove theorem A. In §3, we prove theorem B. In §4, we derive a monotonicity inequality for stationary intrinsic biharmonic maps and prove theorem C.

§2. Proof of theorem A

In this section, we give a proof of theorem A. The ideas are based on: (1) rewritings of the Euler-Lagrange equation for biharmonic maps, which are different from that of [CWY]; (2) applications of the estimates of Riesz potentials in Morrey spaces, due to Adams [A]. First, we recall

Lemma 2.1. *(a) If* $u \in W^{2,2}(\Omega, S^{k-1})$ *is an extrinsic biharmonic map, then it satisfies, in the sense of distributions,*

$$\Delta^2 u = -[|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u\rangle]u \tag{2.1}$$

(b)If $u \in W^{2,2}(\Omega, S^{k-1})$ *is an intrinsic biharmonic map, then it satisfies, in the sense of distributions,*

$$\Delta^2 u + 2\nabla \cdot (|\nabla u|^2 \nabla u) = -\lambda u, \tag{2.2}$$

where $\nabla \cdot$ *is the divergence on* R^m *and*

$$\lambda = [|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u\rangle] + 2|\nabla u|^4.$$

Proof. One can see Proposition 1.1 of [CWY] for (a) and §2.2 of [K] for (b). We omit the detail here. ■

Now we derive an equivalent Euler-Lagrange equation for biharmonic maps as follows. The readers can refer to Chen [C], Shatah [S] for harmonic maps into spheres. Let $\times$ denote the cross product in R^k.

Lemma 2.2. *(a)* $u \in W^{2,2}(\Omega, S^{k-1})$ *is an extrinsic biharmonic map if and only if*

$$\Delta(\nabla \cdot (\nabla u \times u)) = 2\nabla \cdot (\Delta u \times \nabla u) \tag{2.3}$$

(b) $u \in W^{2,2}(\Omega, S^{k-1})$ is an intrinsic biharmonic map if and only if

$$\Delta(\nabla\cdot(\nabla u \times u)) = 2\nabla\cdot(\Delta u \times \nabla u) - 2\nabla\cdot(|\nabla u|^2 \nabla u \times u) \tag{2.4}$$

Proof. For simplicity, we only outline the proof of the eqn. (2.4) and leave (2.3) for the reader. First, observe that the eqn. (2.2) is equivalent to the geometric version:

$$\Delta^2 u + 2\nabla\cdot(|\nabla u|^2 \nabla u) \perp T_u S^{k-1} \tag{2.5}$$

Using the geometry of S^{k-1}, (2.5) is equivalent to

$$(\Delta^2 u + 2\nabla\cdot(|\nabla u|^2 \nabla u)) \times u = 0 \tag{2.6}$$

Observe that

$$\begin{aligned}
\nabla\cdot(|\nabla u|^2 \nabla u) \times u &= \sum_{i=1}^{m} (|\nabla u|^2 u_i)_i \times u \\
&= \sum_{i=1}^{m} (|\nabla u|^2 u_i \times u)_i - \sum_{i=1}^{m} |\nabla u|^2 u_i \times u_i \\
&= \sum_{i=1}^{m} (|\nabla u|^2 u_i \times u)_i \\
&= \nabla\cdot(|\nabla u|^2 \nabla u \times u)
\end{aligned} \tag{2.7}$$

Since

$$\nabla\cdot(\nabla u \times u) = \sum_{i=1}^{m} (u_i \times u)_i = \sum_{i=1}^{m} (u_i \times u_i + u_{ii} \times u) = \Delta u \times u$$

we have

$$\begin{aligned}
\Delta^2 u \times u &= \Delta(\Delta u \times u) - \Delta u \times \Delta u - 2\sum_{i=1}^{m} (\Delta u)_i \times u_i \\
&= \Delta(\nabla\cdot(\nabla u \times u)) - 2\sum_{i=1}^{m} (\Delta u)_i \times u_i \\
&= \Delta(\nabla\cdot(\nabla u \times u)) - 2\sum_{i=1}^{m} (\Delta u \times u_i)_i + 2\Delta u \times \Delta u \\
&= \Delta(\nabla\cdot(\nabla u \times u)) - 2\nabla\cdot(\Delta u \times \nabla u)
\end{aligned} \tag{2.8}$$

Inserting (2.7) and (2.8) into (2.6), we obtain (2.4). ∎

Now we recall both proposition 3.1 and theorem 3.1 of Adams [A] on the estimate of Riesz potentials in Morrey spaces. For this, denote I_α as the Riesz potential operator , i.e. the operator whose convolution kernel is $|x|^{\alpha-m}$, $x \in R^m$.

Proposition 2.3 ([A]). *If $\alpha > 0$, $0 < \lambda \leq m$, $1 < p < \frac{\lambda}{\alpha}$, and $f \in M^{p,\lambda}(R^m) \cap L^p(R^m)$, then*

$$\|I_\alpha(f)\|_{L^{\tilde{p}}(R^m)} \leq C\|f\|_{M^{p,\lambda}(R^m)}^{\frac{\alpha p}{\lambda}} \|f\|_{L^p(R^m)}^{1-\frac{\alpha p}{\lambda}} \tag{2.9}$$

$$\|I_\alpha(f)\|_{M^{\tilde{p},\lambda}(R^m)} \leq C\|f\|_{M^{p,\lambda}(R^m)} \tag{2.10}$$

where $\tilde{p} = \frac{\lambda p}{\lambda - \alpha p}$.

We are ready to prove theorem A now.

Proof of theorem A. For simplicity again, we only sketch the proof for intrinsic biharmonic maps. By considering the intrinsic biharmonic map $u_{x,r}(y) = u(x+ry), y \in B_1$, we may assume, without loss of generalities, that $x = 0$ and $r = 1$. Hence the condition (1.1) becomes

$$\|\nabla u\|_{M^{4,4}(B_1)} + \|\nabla^2 u\|_{M^{2,4}(B_1)} \leq C\epsilon_0^2 \tag{2.11}$$

Let $\tilde{u}$ be an extension of u to R^m such that

$$\|\nabla \tilde{u}\|_{L^4(R^m)} \leq C\|\nabla u\|_{L^4(B_1)} \tag{2.12}$$

and

$$\begin{aligned} \|\nabla \tilde{u}\|_{M^{4,4}(R^m)} + \|\nabla^2 \tilde{u}\|_{M^{2,4}(R^m)} &\leq C(\|\nabla u\|_{M^{4,4}(B_1)} + \|\nabla^2 u\|_{M^{2,4}(B_1)}) \\ &\leq C\epsilon_0^2 \end{aligned} \tag{2.13}$$

Let G be the fundamental solution of Δ^2 on R^m, i.e.

$$\begin{aligned} G(x) &= c_m |x|^{4-m}, \ 0 \neq x \in R^m, \text{ for } m \geq 5 \\ &= c_4 \ln |x|, \quad 0 \neq x \in R^4 \end{aligned}$$

where $c_m \neq 0$ is an universal constant. Now, we define an auxiliary functions $F \in W^{1,2}(R^m)$ by

$$F(x) = \int_{R^m} G(x-y) \nabla \cdot (\Delta \tilde{u} \times \nabla \tilde{u} - |\nabla \tilde{u}|^2 \nabla \tilde{u} \times \tilde{u})(y)\, dy$$

Then we have that, for $1 \leq i \leq m$,

$$\frac{\partial F}{\partial x_i}(x) = -\int_{R^m} \sum_{j=1}^m \frac{\partial^2}{\partial x_i x_j} G(x-y)(\Delta \tilde{u} \times \tilde{u}_j - |\nabla \tilde{u}|^2 \tilde{u}_j \times \tilde{u})(y)\, dy$$

Hence

$$|\nabla F|(x) \le C\int_{R^m} |x-y|^{2-m}(|\Delta\tilde u||\nabla \tilde u| + |\nabla\tilde u|^3)(y)\,dy$$
$$= C(I_2(|\Delta\tilde u||\nabla\tilde u|)(x) + I_2(|\nabla\tilde u|^3)(x))$$

By the Hölder inequality, (2.13) implies that $|\Delta\tilde u||\nabla\tilde u|, |\nabla\tilde u|^3 \in M^{\frac43,4}(R^m)$ and

$$\||\Delta\tilde u||\nabla\tilde u|\|_{M^{\frac43,4}(R^m)} \le \|\Delta\tilde u\|_{M^{2,4}(R^m)}\|\nabla\tilde u\|_{M^{4,4}(R^m)} \tag{2.14}$$

$$\||\nabla\tilde u|^3\|_{M^{\frac43,4}(R^m)} \le \|\nabla\tilde u\|^3_{M^{4,4}(R^m)} \tag{2.15}$$

Now, applying proposition 2.3 with $p=\frac43$, $\alpha=2$, and $\lambda=4$, we have $|\nabla F|\in M^{4,4}(R^m)$ and

$$\|\nabla F\|_{M^{4,4}(R^m)} \le C(\||\Delta\tilde u||\nabla\tilde u|\|_{M^{\frac43,4}(R^m)} + \||\nabla\tilde u|^3\|_{M^{\frac43,4}(R^m)}) \tag{2.16}$$

Therefore we have

$$\|\nabla F\|_{M^{4,4}(R^m)} \le C(\|\Delta u\|_{M^{2,4}(B_1)}\|\nabla u\|_{M^{4,4}(B_1)} + \|\nabla\tilde u\|^3_{M^{4,4}(R^m)})$$
$$\le C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)} \tag{2.17}$$

Now we consider the Hodge decomposition of the 1-form $d\tilde u\times\tilde u = \sum_{i=1}^m \tilde u_i \times \tilde u\, dx_i \in L^4(R^m,\Lambda(R^m))$. It is well-known (cf. [IM]) that there exist a $\Phi\in W^{1,4}(R^m)$ and a 2-form $\Psi\in W^{1,4}(R^m,\Lambda^2(R^m))$ such that

$$d\tilde u\times\tilde u = d\Phi + d^*\Psi,\ d\Psi = 0,\ \text{in } R^m \tag{2.18}$$

and

$$\|\nabla\Phi\|_{L^4(R^m)} + \|\nabla\Psi\|_{L^4(R^m)} \le C\|\nabla\tilde u\|_{L^4(R^m)} \tag{2.19}$$

Taking both sides of the eqn. (2.18) by d^*, we have

$$\Delta^2\Phi = \Delta(\nabla\cdot(\nabla\tilde u\times\tilde u)),\ \text{in } R^m$$

so that

$$\Delta^2\Phi = 2\nabla\cdot(\Delta u\times\nabla u - |\nabla u|^2\nabla u\times u),\ \text{in } B_1 \tag{2.20}$$

On the other hand, it follows from the formula of F that

$$\Delta^2 F = \nabla\cdot(\Delta\tilde u\times\tilde u) - \nabla\cdot(|\nabla\tilde u|^2\nabla\tilde u\times\tilde u),\ \text{in } R^m$$

Therefore, we have

$$\Delta^2(\Phi - 2F) = 0,\ \text{in } B_1. \tag{2.21}$$

Hence, the standard estimate for biharmonic functions implies

$$\|\nabla(\Phi - 2F)\|_{M^{4,4}(B_\theta)} \leq C\theta\|\nabla(\Phi - 2F)\|_{M^{4,4}(B_1)} \tag{2.22}$$

This, combined with (2.17), yields

$$\|\nabla\Phi\|_{M^{4,4}(B_\theta)} \leq C(\theta + \epsilon_0)\|\nabla u\|_{M^{4,4}(B_1)} \tag{2.23}$$

For Ψ, by taking both sides of the eqn. (2.18) by d, we have

$$\Delta\Psi = d\tilde{u} \times d\tilde{u}, \text{ in } R^m. \tag{2.24}$$

Therefore, we have $|\nabla\Psi|(x) \leq CI_1(|\nabla\tilde{u}|^2)(x)$, for $x \in R^m$. Applying the proposition 2.3 again, we obtain

$$\begin{aligned} \|\nabla\Psi\|_{M^{4,4}(R^m)} &\leq C\||\nabla\tilde{u}|^2\|_{M^{2,4}(R^m)} \\ &\leq C\|\nabla\tilde{u}\|^2_{M^{4,4}(R^m)} \\ &\leq C\epsilon_0\|\nabla u\|_{M^{4,4}(B_1)} \end{aligned} \tag{2.25}$$

Putting (2.23) and (2.25) together, we have

$$\|\nabla u \times u\|_{M^{4,4}(B_\theta)} \leq C(\epsilon_0 + \theta)\|\nabla u\|_{M^{4,4}(B_1)} \tag{2.26}$$

This, combined with the fact that $|u| = 1$ implies $|\nabla u| = |\nabla u \times u|$, yields the inequality (1.2) by choosing both $\theta_0 = \theta$ and ϵ_0 sufficiently small. Therefore the proof of theorem A is complete. ■

§3. Proof of theorem B

This section is devoted to the proof of the energy quantization theorem B. The idea is to control the L^2 norm of $\nabla^2 u_n$ over the neck region by the interpolation between the Lorentz space norms $L^{2,\infty}$ and $L^{2,1}$ of $\nabla^2 u_n$, similar to that of [LR] in the context of harmonic maps and [R] in the context of Yang-Mills equations. Since one can easily extend the proof of theorem A from an euclidean domain Ω to a Riemannian manifold (M, g), we first have

Lemma 3.1. *Under the same assumptions of theorem B. There exists an $\epsilon_0 > 0$ such that if, for $B_r(x) \subset M$,*

$$\int_{B_r(x)} |\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4 \, dv_g \leq \epsilon_0^2 \tag{3.1}$$

then, after taking another subsequences,

$$u_n \to u \text{ in } C^4(B_{\frac{r}{2}}(x), S^{k-1})$$

Proof. Let $\epsilon_0 > 0$ be the same constant as in (1.1). Then, it follows from theorem A and the higher order regularity theorem 5.1 of [CWY] that we have, for some $0 < \alpha_0 < 1$,

$$\|u_n\|_{C^{4,\alpha_0}(B_{\frac{r}{2}}(x))} \leq C(\epsilon_0) \tag{3.2}$$

Hence, by taking possible subsequences, we may assume that $u_n \to u$ in $C^4(B_{\frac{r}{2}}(x), S^{k-1})$. ■

Lemma 3.2. *Under the same assumptions as theorem A. There exists a set $\Sigma \subset M$ of finite many points such that $u_n \to u$ in $W^{2,2}_{loc} \cap C^4_{loc}(M \setminus \Sigma, S^{k-1})$. Moreover, $u \in C^\infty(M, S^{k-1})$ is an extrinsic (or intrinsic, respectively) biharmonic map.*

Proof. Define

$$\Sigma = \cap_{r>0}\{x \in M : \liminf_n \int_{B_r(x)} (|\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4)\, dv_g \geq \epsilon_0^2\} \tag{3.3}$$

Then, by the Vitali's covering argument, we have

$$|\Sigma| \leq \frac{1}{\epsilon_0^2} \sup_n \int_M |\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4 = C(\epsilon_0) < \infty$$

For any $x_0 \in M \setminus \Sigma$, it follows from the definition of Σ that there is a $r_0 > 0$ such that

$$\int_{B_{r_0}(x_0)} |\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4 \leq \epsilon_0^2$$

Therefore, Lemma 3.1 implies that $u_n \to u$ in $C^4(B_{\frac{r_0}{2}}(x_0), S^{k-1})$. In particular, $u_n \to u$ in $C^4_{\rm loc} \cap W^{2,2}_{\rm loc}(M \setminus \Sigma, S^{k-1})$. Moreover, $u \in C^\infty(M \setminus \Sigma, S^{k-1}) \cap W^{2,2}(M, S^{k-1})$ is an extrinsic (or intrinsic, respectively) biharmonic map, which can be extended to be in $C^\infty(M, S^{k-1})$ by theorem A. ■

Lemma 3.3. *Under the same notations as above. For any $x_0 \in \Sigma$, there exist $x_n \to x_0$, $r_n \to 0$, and an extrinsic (or intrinsic, respectively) quasi-biharmonic S^4, ω_0, such that*

$$\bar{u}_n(x) \equiv u_n(x_n + r_n x) \to \bar{\omega}_0 \equiv \omega_0 \circ \Pi^{-1}$$

in $C^4_{loc} \cap W^{2,2}_{loc}(R^4, S^{k-1})$

Proof. Since $x_0 \in \Sigma$, there exist $x_n \to x_0$ and $r_n \downarrow 0$ such that

$$\max_{x \in B_{r_0}(x_0)} \{\int_{B_{r_n}(x)} |\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4\} = \frac{\epsilon_0^2}{2} = \int_{B_{r_n}(x_n)} |\nabla^2 u_n|_g^2 + |\nabla u_n|_g^4 \tag{3.4}$$

Define $\bar{u}_n(x) = u_n(x_n + r_n x)$ and $g_n(x) = g(x_n + r_n x)$, for $x \in B_{r_0 r_n^{-1}}$. Then, it is easy to see that $\bar{u}_n : (B_{r_0 r_n^{-1}}, g_n) \to S^{k-1}$ are extrinsic (or intrinsic, respectively) biharmonic maps satisfying

$$\begin{aligned} \frac{\epsilon_0^2}{2} &= \int_{B_1} |\nabla^2 \bar{u}_n|_{g_n}^2 + |\nabla \bar{u}_n|_{g_n}^4 \\ &\geq \int_{B_1(y)} |\nabla^2 \bar{u}_n|_{g_n}^2 + |\nabla \bar{u}_n|_{g_n}^4, \ \forall y \in B_{r_0 r_n^{-1}} \end{aligned} \tag{3.5}$$

Note that $g_n \to g_0$ in $C^4_{\text{loc}}(R^4)$, here g_0 is the euclidean metric in R^4. Therefore, applying Lemma 3.1, we conclude

$$\|\bar{u}_n\|_{C^{4,\alpha_0}(B_1(y))} \leq C_0, \ \forall y \in R^4$$

Hence we may assume that there exists an extrinsic (or intrinsic, respectively) biharmonic map $\bar{\omega} \in C^\infty(R^4, S^{k-1})$ such that

$$\bar{u}_n \to \bar{\omega}, \text{ in } C^4_{\text{loc}} \cap W^{2,2}_{\text{loc}}(R^4, S^{k-1})$$

Moreover, we have

$$\frac{\epsilon_0^2}{2} \leq \int_{R^4} |\nabla^2 \bar{\omega}|^2 + |\nabla \bar{\omega}|^4 < \infty$$

Hence $\omega = \bar{\omega} \circ \Pi$ is an extrinsic (or intrinsic, respectively) quasi-biharmonic S^4, provided that we can prove that ω can be extended smoothly over $(0', 1)$ For this, we need

Lemma 3.4. *Let $\omega \in C^\infty \cap W^{2,2}(R^4, S^{k-1})$ be a nontrivial extrinsic (or intrinsic, respectively) biharmonic map. Then $\bar{\omega} \equiv \omega \circ \Pi \in C^\infty(S^4, S^{k-1})$ is an extrinsic (or intrinsic, respectively) quasi-biharmonic S^4.*

Proof. For simplicity, we assume that ω is an extrinsic biharmonic map. Then $\bar{\omega} \in W^{2,2}(S^4, S^{k-1}) \cap C^\infty(S^4 \setminus \{(0', 1)\}, S^{k-1})$ satisfies

$$-\Delta_{S^4}(-\Delta_{S^4} + 2)\bar{\omega} \perp T_{\bar{\omega}} S^{k-1} \tag{3.6}$$

Since, for any $\epsilon_0 > 0$, there is a $r_0 > 0$ such that

$$\int_{B_{r_0}(0',1) \cap S^4} |\nabla^2_{S^4} \bar{\omega}|^2 + |\nabla_{S^4} \bar{\omega}|^4 \leq \epsilon_0^2$$

Modifying the proof of theorem A, we have that $\bar{\omega} \in C^\infty(B_{\frac{r_0}{2}}(0', 1) \cap S^4, S^{k-1})$. This completes the proof of Lemma 3.4. ■

By extracting all possible extrinsic (or intrinsic, respectively) quasi-biharmonic S^4's, $\{\omega_i\}_{i=1}^d$, with $d \leq C\epsilon_0^{-2}$, we have

$$\lim_n \int_M |\nabla^2 u_n|_g^2 \, dv_g \geq \int_M |\nabla^2 u|^2 \, dv_g + \sum_{i=1}^d \int_{R^4} |\nabla^2 \bar{\omega}_i|^2 \tag{3.7}$$

where $\bar{\omega}_i = \omega_i \circ \Pi^{-1}$ for $1 \le i \le d$. By an induction argument similar to that of [DT] in the context of harmonic maps from surfaces, it suffices to prove that equality holds for (3.7) in the case $d = 1$. Moreover, we may assume that $M = B^4$ and $u_n \to u$ in $C^4_{\text{loc}} \cap W^{2,2}_{\text{loc}}(B^4 \setminus \{0\}, S^{k-1})$ and there exist $r_n \to 0$ such that $u_n(r_n \cdot) \to \bar{\omega}_1$ in $C^4 \cap W^{2,2}_{\text{loc}}(R^4, S^{k-1})$. We need to prove

$$\lim_{n\to\infty} \int_{B^4_\delta \setminus B^4_{Rr_n}} |\nabla u_n|^4 + |\nabla^2 u_n|^2 = o(\delta, R^{-1}) \tag{3.8}$$

where $\lim_{\delta \to 0, R\to\infty} o(\delta, R^{-1}) = 0$. It also follows from the assumption $d = 1$ that for any $\epsilon > 0$, there exists $R >> 1$ such that

$$\int_{B^4_{2r} \setminus B^4_r} |\nabla^2 u_n|^2 + |\nabla u_n|^4 \le \epsilon^2, \ \forall R r_n \le r \le \delta \tag{3.9}$$

This and theorem A then yield

$$|x||\nabla u_n|(x) + |x|^2 |\nabla^2 u_n|(x) \le C\epsilon, \forall 2Rr_n \le |x| \le \frac{\delta}{2} \tag{3.10}$$

Now we let $\tilde{u}_n : R^4 \to R^k$ be an extension of u_n, with compact supports, such that

$$\int_{R^4} |\nabla^2 \tilde{u}_n|^2 + |\nabla \tilde{u}_n|^4 \le C \int_{B^4_\delta} |\nabla^2 u_n|^2 + |\nabla u_n|^4 \tag{3.11}$$

and

$$|x||\nabla \tilde{u}_n|(x) + |x^2|\nabla^2 \tilde{u}_n|(x) \le C\epsilon, \forall x \in B^4_\delta \setminus B^4_{Rr_n}. \tag{3.12}$$

By applying the basic facts on Lorentz spaces (cf. [Z]), we have that $\nabla^2 \tilde{u}_n \in L^{2,\infty}(B^4_\delta \setminus B^4_{Rr_n})$ and

$$\|\nabla^2 \tilde{u}_n\|_{L^{2,\infty}(B^4_\delta \setminus B^4_{Rr_n})} \le C\epsilon \tag{3.13}$$

Now, we proceed to bound the $L^{2,1}$-norm of $\nabla^2 u_n$ as follows. For simplicity, we consider only the case that u_n are intrinsic biharmonic maps so that we have, in B^4_δ,

$$\Delta \nabla \cdot (\nabla u_n \times u_n) = 2\nabla \cdot (\Delta u_n \times \nabla u_n - |\nabla u_n|^2 \nabla u_n \times u_n) \tag{3.14}$$

Define

$$F_n(x) = c_4 \int_{R^4} \ln|x - y| \nabla \cdot (\Delta \tilde{u}_n \times \tilde{u}_n - |\nabla \tilde{u}_n|^2 \nabla \tilde{u}_n \times \tilde{u}_n)(y)\, dy$$

Then, by the Calderon-Zygmund estimate in Lorentz spaces (cf. [SW]), we have

$$\||\nabla^3 F_n|\|_{L^{\frac{4}{3},1}(R^4)} \le C \||\Delta \tilde{u}_n||\nabla \tilde{u}_n| + |\nabla \tilde{u}_n|^3\|_{L^{\frac{4}{3},1}(R^4)} \tag{3.15}$$

Using the Sobolev embedding theorem into Lorentz spaces (see, e.g. [T] or [Z]), we have that $\nabla \tilde{u}_n \in L^{4,2}(R^4)$ and

$$\|\nabla \tilde{u}_n\|_{L^{4,2}(R^4)} \leq \|\nabla^2 \tilde{u}_n\|_{L^{2,2}(R^4)}$$

Hence, by the multiplication theorem between Lorentz spaces, we have

$$\begin{aligned} \||\Delta \tilde{u}_n||\nabla \tilde{u}_n|\|_{L^{\frac{4}{3},1}(R^4)} &\leq \|\Delta \tilde{u}_n\|_{L^{2,2}(R^4)} \|\nabla \tilde{u}_n\|_{L^{4,2}(R^4)} \\ &\leq \|\nabla^2 \tilde{u}_n\|^2_{L^2(R^4)} \end{aligned}$$

and

$$\begin{aligned} \||\nabla \tilde{u}_n|^3\|_{L^{\frac{4}{3},1}(R^4)} &\leq \||\nabla \tilde{u}_n|^2\|_{L^{2,2}(R^4)} \|\nabla \tilde{u}_n\|_{L^{4,2}(R^4)} \\ &\leq \|\nabla \tilde{u}_n\|^2_{L^4(R^4)} \|\nabla^2 \tilde{u}_n\|_{L^2(R^4)} \end{aligned}$$

Putting these estimates together, we have

$$\|\nabla^3 F_n\|_{L^{\frac{4}{3},1}(R^4)} \leq C \|\nabla^2 \tilde{u}_n\|_{L^2(R^4)} (\|\nabla^2 \tilde{u}_n\|_{L^2(R^4)} + \|\nabla \tilde{u}_n\|^2_{L^4(R^4)}) \tag{3.16}$$

Using the Sobolev embedding theorem in Lorentz spaces again, we conclude that $\nabla^2 F_n \in L^{2,1}(R^4)$ and

$$\begin{aligned} &\|\nabla^2 F_n\|_{L^{2,1}(R^4)} \leq C \|\nabla^3 F_n\|_{L^{\frac{4}{3},1}(R^4)} \\ &\leq C(\|\nabla^2 \tilde{u}_n\|^2_{L^2(R^4)} + \|\nabla^2 \tilde{u}_n\|_{L^2(R^4)} \|\nabla \tilde{u}_n\|^2_{L^4(R^4)}) \end{aligned} \tag{3.17}$$

Now using the same Hodge decomposition as in the proof of theorem A, we have that there are $\Phi_n \in W^{1,2}(R^4)$ and a two form $\Psi_n \in W^{1,2}(R^4, \Lambda^2(R^4))$ such that

$$d\tilde{u}_n \times \tilde{u}_n = d\Phi_n + d^* \Psi_n, \ d\Psi_n = 0, \ \text{in } R^4 \tag{3.18}$$

Since

$$\Delta^2 (\Phi_n - 2F_n) = 0, \ \text{in } B^4_\delta$$

we have, by the standard estimate for biharmonic functions,

$$\begin{aligned} \|\nabla^2 (\Phi_n - 2F_n)\|_{L^{2,1}(B^4_{\frac{\delta}{2}})} &\leq C \|\nabla^2 (\Phi_n - 2F_n)\|_{L^2(B^4_\delta)} \\ &\leq C \|\nabla^2 \tilde{u}_n\|_{L^2(B^4_\delta)} \end{aligned}$$

For Ψ_n, we have

$$\Delta \Psi_n = d\tilde{u}_n \times d\tilde{u}_n, \ \text{in } R^4 \tag{3.19}$$

Hence

$$\|\nabla^2\Psi_n\|_{L^{2,1}(R^4)} \le C\||\nabla\tilde{u}_n|^2\|_{L^{2,1}(R^4)} \le C\|\nabla\tilde{u}_n\|^2_{L^{4,2}(R^4)}$$
$$\le C\|\nabla^2\tilde{u}_n\|_{L^2(R^4)} \le C\|\nabla^2 u_n\|_{L^2(B^4_\delta)}$$

Putting all these estimates together, we obtain

$$\|\nabla(\nabla u_n \times u_n)\|_{L^{2,1}(B^4_{\frac{\delta}{2}})} \le C\|\nabla^2 u_n\|_{L^2(B^4_\delta)} \le C$$

On the other hand, since

$$|\nabla^2 u_n| \le |\nabla(\nabla u_n \times u_n)| + |\nabla u_n|^2$$

we have

$$\|\nabla^2 u_n\|_{L^{2,1}(B^4_\delta)} \le \|\nabla(\nabla u_n \times u_n)\|_{L^{2,1}(B^4_\delta)} + \|\nabla u_n\|^2_{L^{4,2}(B^4_\delta)}$$
$$\le C(\|\nabla^2 u_n\|_{L^2(B^4_\delta)} + \|\nabla u_n\|_{L^4(B^4_\delta)}) \le C$$

Therefore, by the duality between $L^{2,1}$ and $L^{2,\infty}$, we have

$$\|\nabla^2 u_n\|^2_{L^2(B^4_{\frac{\delta}{2}}\setminus B^4_{Rr_n})} \le \|\nabla^2 u_n\|_{L^{2,1}(B^4_\delta)}\|\nabla^2 u_n\|_{L^{2,\infty}(B^4_\delta\setminus B^4_{Rr_n})} \le C\epsilon \tag{3.20}$$

Now we apply the Nirenberg interpolation inequality to get

$$\|\nabla u_n\|^2_{L^4(B^4_\delta\setminus B^4_{Rr_n})} \le C\|u_n\|_{L^\infty(B^4_{\frac{\delta}{2}})}(\||\nabla u_n| + |\nabla^2 u_n|\|_{L^2(B^4_{\frac{\delta}{2}}\setminus B^4_{Rr_n})})$$
$$\le C\epsilon + o(n^{-1})$$

Here we have used the fact that

$$\|\nabla u_n\|_{L^2(B^4_\delta} = \|\nabla u\|_{L^2(B^4_\delta)} + o(n^{-1}) = o(\delta, n^{-1})$$

Therefore, we finish the proof of theorem B. ■

§4. Proof of theorem C

In this section, we first derive an inequality of monotonicity type for the class of stationary intrinsic biharmonic maps, which includes smooth intrinsic biharmonic maps. Then, applying a slightly weaker estimate of the eqn. (2.4) and a variance of the iteration scheme of §4 [CWY], we show that away from a closed set of $(m-4)$-dimensional Hausdorff measure zero the condition (1.1) of theorem A is satisfied so that the conclusion of theorem C is true.

Lemma 4.1. *If $u \in W^{2,2}(\Omega, S^{k-1})$ is a stationary intrinsic biharmonic map. Then, for any $X \in C_0^1(\Omega, R^m)$, we have*

$$\int_\Omega X(|\Delta u|^2) + (\nabla \cdot X)|\Delta u|^2 = \int_\Omega 4|\nabla u|^2 \sum_{ij=1}^m u_i u_j X_j^i - |\nabla u|^4 \nabla \cdot X \tag{4.1}$$

Proof. Denote $u_t(x) = u(x + tX(x))$. First, observe that

$$|(\Delta u_t)^T|^2 = |\Delta u_t|^2 - |\nabla u_t|^4$$

Therefore, the stationarity of u implies

$$\frac{d}{dt}|_{t=0} \int_\Omega |\Delta u_t|^2 = \frac{d}{dt}|_{t=0} \int_\Omega |\nabla u_t|^4 \tag{4.2}$$

As in Lemma 3.1 of [CWY], we have

$$\frac{d}{dt}|_{t=0} \int_\Omega |\Delta u_t|^2 = \int_\Omega X(|\Delta u|^2) + (\nabla \cdot X)|\Delta u|^2$$

For the right hand side of (4.2), we have

$$\frac{d}{dt}|_{t=0} \int_\Omega |\nabla u_t|^4 = \int_\Omega 4|\nabla u|^2 \sum_{ij} u_i u_j X_j^i - |\nabla u|^4 \nabla \cdot X$$

Putting these two identities together, we prove (4.1). ■

Lemma 4.2. *Suppose that $u \in W^{2,2}(\Omega, S^{k-1})$ is a stationary intrinsic biharmonic map. Then, for any $x \in \Omega$ and $0 < r < R < dist(x, \partial\Omega)$, it holds*

$$\begin{aligned} &R^{4-m} \int_{B_R(x)} |\Delta u|^2 - r^{4-m} \int_{B_r(x)} |\Delta u|^2 \\ &\geq 2 \int_{\partial B_R(x)} \left(-\frac{x_i u_l u_{il}}{|x|^{m-2}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) dH^{m-1} \\ &- 2 \int_{\partial B_r(x)} \left(-\frac{x_i u_l u_{il}}{|x|^{m-2}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) dH^{m-1} \end{aligned} \tag{4.3}$$

Proof. For simplicity, we assume that $x = 0 \in \Omega$. Let $X = \sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$. First, observe that by choosing suitable cut-off functions, (4.1) implies that for a.e. $r > 0$,

$$\begin{aligned} &\int_{B_r} X(|\Delta u|^2) + m|\Delta u|^2 - r \int_{\partial B_r} |\Delta u|^2 \\ &= \int_{B_r} (4-m)|\nabla u|^4 + r \int_{\partial B_r} |\nabla u|^4 - 4r \int_{\partial B_r} |\nabla u|^2 |\frac{\partial u}{\partial r}|^2 \end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$\begin{aligned}
r^{m-3}\frac{d}{dr}(r^{4-m}\int_{B_r}|\Delta u|^2) &= r\int_{\partial B_r}|\Delta u|^2 + (4-m)\int_{B_r}|\Delta u|^2 \\
&= \int_{B_r}(X(|\Delta u|^2)+4|\Delta u|^2) + (m-4)\int_{B_r}|\nabla u|^4 \\
&- r\int_{\partial B_r}|\nabla u|^4 + 4r\int_{\partial B_r}|\nabla u|^2|\frac{\partial u}{\partial r}|^2
\end{aligned}$$

Note that the eqn. (2.5) yields that

$$x_i u_i \Delta^2 u = -2x_i u_i \nabla\cdot(|\nabla u|^2\nabla u), x\in\Omega$$

Therefore, we can proceed in the same way as that of proposition 3.2 of [CWY] to estimate

$$\begin{aligned}
&\int_{B_r}(X(|\Delta u|^2)+4|\Delta u|^2) \\
&= \int_{B_r}(2x_i(\Delta u)_i(\Delta u)+4|\Delta u|^2) \\
&= \int_{\partial B_r}\frac{2x_ix_ku_{ik}\Delta u - 2x_ix_ku_i(\Delta u)_k}{r} \\
&+ \int_{B_r}(2|\Delta u|^2 + 2(\Delta u)_k u_k - 4x_iu_i\nabla\cdot(|\nabla u|^2\nabla u)) \\
&= \int_{\partial B_r}\frac{2x_ix_ku_{ik}\Delta u - 2x_ix_ku_i(\Delta u)_k + 2x_ku_k\Delta u}{r} \\
&+ 4\int_{B_r}(x_iu_i)_j|\nabla u|^2u_j - 4r\int_{\partial B_r}|\nabla u|^2|\frac{\partial u}{\partial r}|^2 \\
&= \int_{\partial B_r}\frac{2x_ix_ku_{ik}\Delta u - 2x_ix_ku_i(\Delta u)_k + 2x_ku_k\Delta u}{r} \\
&+ (4-m)\int_{B_r}|\nabla u|^4 + r\int_{\partial B_r}|\nabla u|^4 - 4r\int_{\partial B_r}|\nabla u|^2|\frac{\partial u}{\partial r}|^2
\end{aligned}$$

Inserting this identity into the above identity, we obtain

$$r^{m-3}\frac{d}{dr}(r^{4-m}\int_{B_r}|\Delta u|^2) = \int_{\partial B_r}\frac{2x_ix_ku_{ik}\Delta u - 2x_ix_ku_i(\Delta u)_k + 2x_ku_k\Delta u}{r}$$

Now we can follow exactly the arguments (3.3)–(3.8) of [CWY] to get (4.3). ■

In order to apply Lemma 4.2 to verify that the condition (1.1) holds for H^{m-4} a.e. $x\in\Omega$. We need an estimate, similar to Lemma 4.5 of [CWY], for the eqn. (2.4). More precisely,

Lemma 4.3. *Let $u \in W^{2,2}(\Omega, S^{k-1})$ be an intrinsic biharmonic map. Then, for any $0 < \theta < 1$ and $B_r \subset \Omega$, we have, for $p = \frac{4m}{3m-4} > \frac{4}{3}$ and $q = \frac{4m}{m+4} < 4$,*

$$((\theta r)^{p-m} \int_{B_{\theta r}} |\nabla u|^p)^{\frac{1}{p}} \leq C\theta^{1-\frac{m}{p}} [(r^{4-m} \int_{B_r} |\Delta u|^2)^{\frac{1}{2}} + (r^{4-m} \int_{B_r} |\nabla u|^4)^{\frac{1}{4}}] \cdot (r^{q-m} \int_{B_r} |\nabla u|^q)^{\frac{1}{q}} + C\theta (r^{p-m} \int_{B_r} |\nabla u|^p)^{\frac{1}{p}} \tag{4.5}$$

Proof. Since each term of (4.5) is invariant under scalings, it suffices to prove it for $r = 1$. We use the same ideas as in the proof of theorem, but replacing the estimates for Riesz potentials in Morrey spaces by that in L^p spaces. With the same notations as in theorem A, we have

$$(\theta^{p-m} \int_{B_\theta} |\nabla u|^p)^{\frac{1}{p}} \leq C(\theta^{p-m} \int_{B_\theta} |\nabla(\Phi - 2F)|^p)^{\frac{1}{p}} + C\theta^{1-\frac{m}{p}} (\int_{R^m} |\nabla F|^p + |\nabla \Psi|^p)^{\frac{1}{p}}$$

Since $\Phi - 2F$ is biharmonic function in B_1, we have

$$(\theta^{p-m} \int_{B_\theta} |\nabla(\Phi - 2F)|^p)^{\frac{1}{p}} \leq C\theta (\int_{B_1} |\nabla(\Phi - 2F)|^p)^{\frac{1}{p}} \leq C\theta (\int_{B_1} |\nabla u|^p)^{\frac{1}{p}}$$

Since $|\nabla F| \leq C(I_2(|\Delta \tilde{u}||\nabla \tilde{u}|) + I_2(|\nabla \tilde{u}|^3))$, $|\nabla \Psi|(x) \leq CI_1(|\nabla \tilde{u}|^2)(x)$, and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q} - \frac{2}{m}$, we have

$$\begin{aligned} \|\nabla F\|_{L^p(R^m)} &\leq C(\|\Delta \tilde{u}\|_{L^2(R^m)} \|\nabla \tilde{u}\|_{L^q(R^m)} + \|\nabla \tilde{u}\|^2_{L^4(R^m)} \|\nabla \tilde{u}\|_{L^q(R^m)}) \\ &\leq C(\|\Delta u\|_{L^2(B_1)} \|\nabla u\|_{L^q(B_1)} + \|\nabla u\|^2_{L^4(B_1)} \|\nabla u\|_{L^q(B_1)}) \end{aligned}$$

$$\begin{aligned} \|\nabla \Psi\|_{L^p(R^m)} &\leq C\|\nabla \tilde{u}\|_{L^4(R^m)} \|\nabla \tilde{u}\|_{L^2(R^m)} \\ &\leq C\|\nabla u\|_{L^4(B_1)} \|\nabla u\|_{L^2(B_1)} \leq C\|\nabla u\|_{L^4(B_1)} \|\nabla u\|_{L^q(B_1)} \end{aligned}$$

Putting these estimates together, we prove (4.5). ■

Now we are ready to control the local Morrey norms of ∇u and $\nabla^2 u$ near a point, as in (1.1), where the normalized $W^{2,2}$-norm is small. More precisely,

Lemma 4.4. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, 1)$ such that if $u \in W^{2,2}(\Omega, S^{k-1})$ is a stationary intrinsic biharmonic map satisfying, for $B_r \subset \Omega$,*

$$r^{4-m} \int_{B_r} |\nabla^2 u|^2 + r^{2-m} \int_{B_r} |\nabla u|^2 \leq \epsilon_0^2 \tag{4.6}$$

then

$$\|\nabla u\|_{M^{4,4}(B_{\theta_0 r})} + \|\nabla^2 u\|_{M^{2,4}(B_{\theta_0 r})} \leq C\epsilon_0 \tag{4.7}$$

Proof. We first observe that, by an interpolation inequality of Nirenberg [N], we have, for all $2 \leq q \leq 4$,

$$\begin{aligned}(r^{q-m}\int_{B_r} |\nabla u|^q)^{\frac{1}{q}} &\leq (r^{4-m}\int_{B_r} |\nabla u|^4)^{\frac{1}{4}} \\ &\leq C\|u\|^{\frac{1}{2}}_{L^\infty(B_r)}(r^{4-m}\int_{B_r} |\nabla^2 u|^2 + r^{2-m}\int_{B_r} |\nabla u|^2)^{\frac{1}{2}} \\ &\leq C\epsilon_0 \end{aligned} \tag{4.8}$$

Now we can follow exactly the proof of Lemma 4.8 in [CWY], with Lemma 4.5 of [CWY] replaced by Lemma 4.3 above, to establish (4.7). We omit the detail here. ∎

Proof of theorem C. Define

$$\Sigma = \{x \in \Omega : \limsup_{r\downarrow 0} r^{2-m}(\int_{B_(x)} r^2|\nabla^2 u|^2 + |\nabla u|^2) \geq \epsilon_0^2\}$$

where $\epsilon_0 > 0$ is the same constant as in (4.6). We first observe that (4.8) implies $\nabla u \in L^4(\Omega)$ so that, we have, by the Hölder inequality, $\Sigma \subset \Sigma_1$, where

$$\Sigma_1 = \{x \in \Omega : \limsup_{r\downarrow 0} r^{4-m}(\int_{B_(x)} |\nabla^2 u|^2 + |\nabla u|^4) \geq \epsilon_0^2\}$$

By Federer-Ziemer (cf. [Z]), we have $H^{m-4}(\Sigma_1) = 0$ so that $H^{m-4}(\Sigma) = 0$ as well. Now, for any $x_0 \in \Omega \setminus \Sigma$, there exists a $r_0 > 0$ such that

$$r_0^{2-m}\int_{B_{r_0}(x_0)} r_0^2|\nabla^2 u|^2 + |\nabla u|^2 < \epsilon_0^2$$

this, combined with Lemma 4.4, implies that there exists a $\delta_0 \in (0,1)$ such that, for any $y \in B_{\frac{r_0}{4}}(x_0)$,

$$\|\nabla u\|_{M^{4,4}(B_{\delta_0 r_0}(y))} + \|\nabla^2 u\|_{M^{2,4}(B_{\delta_0 r_0}(y))} \leq C\epsilon_0 \tag{4.9}$$

Therefore, theorem A and its iterations imply that there exists a $\theta_0 \in (0,1)$ such that, for $r_1 = \delta_0 r_0$ and $l \geq 1$,

$$\|\nabla u\|_{M^{4,4}(B_{\theta_0^l r_1}(y))} \leq 2^{-l}\|\nabla u\|_{M^{4,4}(B_{r_1}(y))}, \forall y \in B_{\frac{r_0}{4}}(x_0) \tag{4.10}$$

This, combined with the Morrey Lemma, implies $u \in C^{\alpha_0}(B_{\frac{r_0}{4}}(x_0), S^{k-1})$ for some $\alpha_0 \in (0,1)$. Therefore, $\Omega \setminus \Sigma$ is open and $u \in C^{\alpha_0}(\Omega \setminus \Sigma, S^{k-1})$. Finally, we have, by the higher order regularity theorem 5.1 of [CWY], $u \in C^\infty(\Omega \setminus \Sigma, S^{k-1})$. ∎

REFERENCES

[A] D. Adams, *A note on Riesz potentials*. Duke Math. J., V. 42, No. 4 (1975) 765-778.

[B] F. Bethuel, *On the singular set of stationary harmonic maps.* Manuscripta Math. 78 (1993), no. 4, 417–443.

[C] Y. M. Chen, *The weak solutions to the evolution problems of harmonic maps.* Math. Z. 201 (1989), no. 1, 69–74.

[CG] J. Coron, R. Gulliver, *Minimizing p-harmonic maps into spheres.* J. Reine Angew. Math. 401 (1989), 82–100.

[CWY] S. Y. A. Chang, L. Wang, P. Yang, *A regularity theory of biharmonic maps.* Comm. Pure Appl. Math. 52 (1999), no. 9, 1113–1137.

[CWY1] S. Y. A. Chang, L. Wang, P. Yang, *Regularity of harmonic maps.* Comm. Pure Appl. Math. 52 (1999), 1099-1111.

[CY] S. Y. A. Chang, P. Yang, *Extremal metrics of zeta function determinants on* 4-*manifolds.* Ann. of Math. (2) 142 (1995), no. 1, 171–212.

[DT] W. Y. Ding, G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces.* Comm. Anal. Geom. 3 (1995), no. 3-4, 543–554.

[E] L. C. Evans, *Partial regularity for stationary harmonic maps into spheres.* Arch. Rational Mech. Anal. 116 (1991), no. 2, 101–113.

[EL] J. Eells, L. Lemaire, Another report on harmonic maps. Bull. London Math. Soc. 20 (1988), no. 5, 385–524

[H] F. Hélein, *Rgularité des applications faiblement harmoniques entre une surface et une variété riemannienne.* C. R. Acad. Sci. Paris Sr. I Math. 312 (1991), no. 8, 591–596.

[IM] T. Iwaniec, G. Martin, *Quasiregular mappings in even dimensions.* Acta Math. 170 (1993), no. 1, 29–81.

[J] J. Jost, Two dimensional geometric variational problems. New York, Wiley, 1991.

[K] Y. Ku, *Interior and boundary regularity of intrinsic biharmonic maps to spheres.* Preprint.

[LR] F. H. Lin, T. Riviére, *Energy quantization for harmonic maps.* Duke Math. J. 111 (2002), no. 1, 177–193.

[LW] F. H. Lin, C. Y. Wang, *Energy identity of harmonic map flows from surfaces at finite singular time.* Calc. Var. Partial Differential Equations 6 (1998), no. 4, 369–380.

[MR] Y. Meyer, T. Riviére, *A partial regularity result for a class of stationary Yang-Mills fields in high dimensions.* To appear.

[N] L. Nirenberg, *On elliptic partial differential equations.* Ann. Acuola Norm. Sup. Pisa Cl. Sci. (4) 13 (1959), 115-162.

[P] T. Parker, *Bubble tree convergence for harmonic maps.* J. Differential Geom. 44 (1996), no. 3, 595–633

[R] T. Riviére, *Interpolation spaces and energy quantization for Yang-Mills fields.* Comm. Anal. Geom. 10 (2002), no. 4, 683–708.

[Q] J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres.* Comm. Anal. Geom. 3 (1995), no. 1-2, 297–315.

[S] J. Shatah, *Weak solutions and development of singularities of the* $\mathbf{SU}(2)$ *σ-model.* Comm. Pure Appl. Math. 41 (1988), no. 4, 459–469

[SU] R. Schoen, K. Uhlenbeck, *A regularity theory for harmonic maps.* J. Diff. Geom. 17 (1982), 307-335.

[SW] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces. Princeton (1971).

[T] L.Tartar, *Imbedding theorems of Sobolev spaces into Lorentz spaces.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 3, 479–500.

[TT] T. Tao, G. Tian, *A singularity removal theorem for Yang-Mills fields in higher dimensions.* Preprint.

[U1] K. Uhlenbeck, *Removable singularities in Yang-Mills fields.* Comm. Math. Phys. 83 (1982), 11-29.

[U2] K. Uhlenbeck, *Connections with* L^p*-bounds on curvature.* Comm. Math. Phys. 83 (1982), 31-42.

[W] C. Y. Wang, *Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets.* Houston J. Math. 22 (1996), no. 3, 559–590.

[W1] C. Y. Wang, *Biharmonic maps from* R^4 *into a Riemannian manifold.* Preprint.

[W2] C. Y. Wang, *Stationary biharmonic maps from* R^m *into a Riemannian manifold* . Preprint.

[Z] W. P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variation. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.