COUNTING INDEPENDENT SETS ON PATH-SCHEMES.

SERGEY KITAEV

ABSTRACT. We give the generating function for the number of independent sets on the class of well-based path-schemes, which generalizes the known result in this direction.

KEYWORDS: Independent sets, path-schemes, generating functions, binary stings avoidance

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{1, 2, \ldots, n\}$ and set of edges E. An *independent set* (also called a *stable set* in the literature) of G is a subset S of V such that no two vertices in S are adjacent. The set of all independent sets of a graph G is denoted by $I(G)$. An independent set is *maximal* if it is not a subset of any larger independent set, and *maximum* if there are no larger independent sets in the graph. The *independence number* $\alpha(G)$ (also called the *stability number*) is the cardinality of a maximum independent set in G.

The two problems of determining maximal and maximum independent sets have received considerable attention, particularly since the computation of the independence number is known to be an NP-complete problem [7]. These problems was extensively studied for various classes of graphs, including trees, forests, (connected) graphs with at most one cycle, bipartite graphs, k-connected graphs, and others (see [6] for a survey). The most efforts were made there to find $\alpha(G)$ for a graph G, and there are not so many works about counting the number of maximum independent sets. However, counting cardinality of $I(G)$, being a very challenging and interesting enumerative combinatorics problem, received even less attention, and very few papers deal with it (see, e.g., [1, 2, 3, 4, 8] and references therein). A motivation for finding $|I(G)|$ is, for instance, the fact that for some classes of graphs, the set

The author thanks the Mathematics Department at the University of Kentucky for its support during the 2003-2004 academic year.

of independent sets $I(G)$ has an interpretation in terms of other combinatorial objects (see [1, 3]). For example, in [3], it was shown that there is a bijection between the set of independent sets of a *symmetric Ferrers graph* on $2n$ vertices and the parts of all compositions (ordered non-empty partitions) of $n+1$.

The main objective in this paper is to obtain the generating function for the number of independent sets on the class of *well-based path-schemes* (see Section 2 for definitions), which generalizes the known result in this direction [8]. We achieve that by reformulating the problem in terms of combinatorics on words, and then applying a known result.

2. Preliminary

Let $V = \{1, 2, \dots, n\}$ and M be a subset of V. A *path-scheme* $P(n, M)$ is a graph $G = (V, E)$, where the edge set E is $\{(x, y) \mid |x-y| \in M\}$.

Note that from the definition, $P(n, M)$ is a simple graph, and thus its adjacency matrix A is symmetric. Moreover, if the columns and rows of A are ordered naturally, that is, node i corresponds to the i-th column and to the i-th row, then for $1 \leq i < j < n$, $A(i, j) = A(i+1, j+1)$, since $|i-j|$ is in M iff $|(i+1)-(j+1)|$ is in M. Thus, we can construct the upper triangular part of A by shifting the first row to the right, and then we use the symmetry to fill in the remain entries of A.

Suppose $k \geq 2$ and $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ is a set of words of the form $A_i = 1\underbrace{0\dots0}_{a_i-1}1$, where $a_i \geq 1$, and $a_i < a_j$ if $i < j$. Moreover, we assume that for any $i > 1$ and $A_i \in \mathcal{A}$, if we replace any number of 0's in A_i by 1's, then we obtain a word A'_i that contains the word $A_j \in \mathcal{A}$ as a subword for some $j < i$. In this case, we call $\mathcal{A}$ *well-based set*, and we call the sequence of a_is associated with $\mathcal{A}$ *well-based sequence*.

Any well-based set must contain the word 11. Indeed, if we replace all 0's by 1's in, say, A_2 then A_1 must be a subword of the obtained word. So, we may extend our definition to the case $k = 1$. We define $\mathcal{A} = \{11\}$ to be a well-based set. We see that any well-based sequence starts from 1, and, clearly, if we take any number of consecutive initial elements of a well-based sequence, we get a well-based sequence. Few examples of well-based sets and associated with them sequences are given in Table 1 (i copies of 0 is denoted by 0^i).

We call a scheme $P(n, M)$ *well-based scheme*, if the elements of M listed in the increasing order form a well-based sequence.

Well-based set	Corresponding well-based sequence
$\{11, 101, 1001, \ldots, 10^{k-1}1\}$	$1, 2, 3, \ldots, k$
$\{11, 1001, 100001, \ldots, 10^{2m}1\}$	$1, 3, 5, \ldots, 2m+1$
$\{11, 101, 1001, 1000001, 10000001\}$	1,2,3,6,7

TABLE 1. Examples of well-based sets and well-based sequences

It is known [8], that $|I(P(n, \{1\}))| = F(n+2)$ and, more generally,

$$|I(P(n, \{1, 2, \ldots, k-1\}))| = F_k(n+k), \tag{1}$$

where $F(n)$ is the n-th Fibonacci number and $F_k(n)$ is the n-th k-*generalized Fibonacci number* defined, in our context, as $F_k(1) = \cdots = F_k(k) = 1$ and $F_k(n) = F_k(n-1) + F_k(n-k)$. Note that in our notation, $F(n) = F_2(n)$.

In Section 3, we find the generating function for the number of independent sets of an arbitrary well-based path-scheme. The known result mentioned above can be extracted from our generating functions, since it corresponds to a well-based sequence $1, 2, \ldots, k-1$.

Before going further, we need some notions and results in a certain area of combinatorics on words which perhaps can be best described as "binary strings and substring avoidance." In our presentation we follow [9], although the original ideas appear in [5].

A *binary string* is a string that consists only of the digits 0 and 1. If $X_1 = a_0a_1 \ldots a_{k-1}$ and $X_2 = b_0b_1 \ldots b_{\ell-1}$ are two binary strings of length k and ℓ respectively, then the *correlation* $c_{12} = c_0c_1 \ldots c_{k-1}$ is the binary string obtained by setting $c_j = 1$ if $a_i = b_{i+j}$ for every $1 \le i \le k-1-j$, and $c_j = 0$ otherwise. For example, if $X_1 = 110$ and $X_2 = 1011$, then $c_{12} = 011$ and $c_{21} = 0010$, as depicted below:

```
          1 1 0                   1 0 1 1
  ---------------------     -------------------
      1 0 1 1      0                1 1 0    0
    1 0 1 1        1              1 1 0      0
  1 0 1 1          1            1 1 0        1
                              1 1 0          0
```

So, in general $c_{ij} \neq c_{ji}$ (they can even be of different lengths). The *autocorrelation* of a word X_1 is just c_{11}, the correlation of X_1 with itself. For instance, if $X_1 = 1011$ then $c_{11} = 1001$. This is convenient to interpret a correlation $c_{ij} = c_0c_1 \ldots c_{k-1}$ as a polynomial $c_{ij}(x) = c_0 + c_1x + \cdots + c_{k-1}x^{k-1}$.

The following theorem is the main tool in our considerations.

Theorem 1. ([9, Th. 24]) *The generating function for the number of binary strings that avoid the substrings $b_1, b_2, \dots, b_n$, of length $\ell_1, \ell_2, \dots, \ell_n$ respectively, none included in any other, is given by the formula*

$$S(x) = \frac{\begin{vmatrix} -c_{11}(x) & \cdots & -c_{1n}(x) \\ \vdots & \ddots & \vdots \\ -c_{n1}(x) & \cdots & -c_{nn}(x) \end{vmatrix}}{\begin{vmatrix} (1-2x) & 1 & \cdots & 1 \\ x^{\ell_1} & -c_{11}(x) & \cdots & -c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ x^{\ell_n} & -c_{n1}(x) & \cdots & -c_{nn}(x) \end{vmatrix}}. \tag{2}$$

3. Main Results

Our main result in this paper is the following theorem.

Theorem 2. *Let $M = \{a_1, a_2, \dots, a_k\}$ be a subset of $V = \{1, 2, \dots, n\}$ such that the sequence $a_1, a_2, \dots, a_k$ is well-based (in particular, $a_1 = 1$). Let $c(x) = 1 + x + \sum_{i=2}^{k} x^{a_i}$. Then the generating function for the number of independent sets on the well-based path-scheme $P = P(n, M)$ (with vertex set V) is given by*

$$G(x) = \frac{c(x)}{(1-x)c(x) - x}.$$

Proof. If x is a vertex in P, we denote by $N(x)$ the set of its neighbors in P. We identify independent sets with the corresponding (0,1)-incidence vectors, indexed by V. These vectors are called *stable vectors* in some literature. Let $S(P)$ denote the set of all stable vectors of P. Then

$$S_n(P) = \{T \in \{0,1\}^n \mid \forall x \in V \ T(x) = 1 \Rightarrow T(y) = 0 \ \forall y \in N(x)\}.$$

Thus, our purpose is equivalent to finding the generating function for $|S_n(P)|$.

Let A be the adjacency matrix of P with rows and columns ordered naturally. One can see that the first row of A has 0's everywhere except for the entries $A(1, a_i + 1)$, where $i = 1, 2, \dots, k$. Indeed, if $A(1, x+1) = 1$, and $x \neq a_i$ for some i, then we must have $x \in M$, contradiction.

Recall that the upper triangular part of A is constructed by shifting the first row to the right, which gives that a vector x belongs to $S_n(P)$ iff x avoids each substring $b_i = 1\underbrace{0\dots0}_{a_i-1}1$ for $i = 1, 2, \dots, k$. Let us prove the last statement.

We first prove necessity. Assume that for a vector $T \in S_n(P)$, $T(j) = T(j + a_i) = 1$ and $T(t) = 0$ for $j < t < j + a_i$ and $1 \leq j \leq n - a_i$. From the way we construct A, $(j + a_i) \in N(j)$. We get a contradiction with the definition of $S_n(P)$.

Let us now prove sufficiency. We need to show that if a vector T does not belong to $S_n(P)$ then it must contain b_s for some s, $1 \leq s \leq k$. A vector T does not belong to $S_n(P)$ if there exist two adjacent nodes, say j and h, $j < h$, such that $T(j) = T(h) = 1$. From the construction of A, we must have $h = j + a_i$ for some i, $1 \leq i \leq k$. If $T(t) = 0$ for all t such that $j < t < h = j + a_i$ then we are done. If some of $T(t)$, for $j < t < h$, are not 0's, T must contain b_s for some s, $1 \leq s < i$ due to the fact, that the sequence of a_is is well-based, and therefore the set $\{b_1, b_2, \ldots, b_k\}$ is well-based (this set is associated with the sequence)[1].

So, $|S_n(P)|$ is given by the number of different binary stings avoiding the substrings b_1, $b_2, \ldots, b_k$, and we may use Theorem 1 since none of b_is is included in any other.

One can easily check that the autocorrelation $c_{ii} = 1 + x^{a_i}$, and for $i < j$, the correlations $c_{ij} = x^{a_i}$ and $c_{ji} = x^{a_j}$. The corresponding lengths are $\ell_i = a_i + 1$, for $1 \leq i \leq k$. Thus (2) in our case is

$$G(x) = \frac{\begin{vmatrix} -(1+x^{a_1}) & -x^{a_1} & \cdots & -x^{a_1} \\ -x^{a_2} & -(1+x^{a_2}) & \cdots & -x^{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ -x^{a_k} & -x^{a_k} & \cdots & -(1+x^{a_k}) \end{vmatrix}}{\begin{vmatrix} (1-2x) & 1 & 1 & \cdots & 1 \\ x^{a_1+1} & -(1+x^{a_1}) & -x^{a_1} & \cdots & -x^{a_1} \\ x^{a_2+1} & -x^{a_2} & -(1+x^{a_2}) & \cdots & -x^{a_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{a_k+1} & -x^{a_k} & -x^{a_k} & \cdots & -(1+x^{a_k}) \end{vmatrix}}.$$

To take the determinant in the numerator, we replace the first row by the sum of all the rows, then factor out some terms from the determinant, and then add to each column the fist one multiplied by (-1) to get

$$(-1)^k \cdot (1 + \sum_{i=1}^{k} x^{a_i}) \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x^{a_2} & 1 & 0 & \cdots & 0 \\ x^{a_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{a_k} & 0 & 0 & \cdots & 1 \end{vmatrix} = (-1)^k \cdot c(x).$$

[1]Note that this is the only place we use the fact that the sequence $a_1, a_2, \ldots, a_k$ is well-based.

To take the determinant in the denominator, we may replace the first column by the sum of the first column and the last column times x; we replace any column i, $1 < i < k+1$ by the sum of column i and the last column times (-1). Finally, we replace the last row by the sum:

$$\frac{x}{1-x} \cdot (\text{row } 1) + (\text{row } 2) + \cdots + (\text{row (k+1)}),$$

to get an upper triangular matrix having the determinant

$$(-1)^k \cdot (1-x) \left(1 - \frac{x}{1-x} + \sum_{i=1}^{k} x^{a_i} \right) = (-1)^k \cdot ((1-x)c(x) - x).$$

Thus the statement is proved. □

Let us discuss some corollaries to Theorem 2.

If $M = \{1, 2, \ldots, k-1\}$ then we can apply our theorem, since the sequence $1, 2, \ldots, k-1$ is well-based. In this case, we get

$$G(x) = \sum_{n \geq 0} g_n x^n = \frac{1 + x + \cdots + x^{k-1}}{1 - x - x^k},$$

and thus, using the form of the generating function, the sequence $g_n = |I(P(n, M))|$ satisfies the recurrence $g_n = g_{n-1} + g_{n-k}$ with $g_{1-k} = g_{2-k} = \cdots = g_0 = 1$, which agrees with (1).

If $M = \{1, 3, 5, 7\}$ then M is well-based. Theorem 2 gives us that

$$G(x) = \sum_{n \geq 0} w_n x^n = \frac{1 + x + x^3 + x^5}{1 - x - x^2 + x^3 - x^4 + x^5 - x^6}.$$

Thus, in this case the sequence w_n satisfies the recurrence formula

$$w_n = w_{n-1} + w_{n-2} - w_{n-3} + w_{n-4} - w_{n-5} + w_{n-6},$$

with the initial conditions: $w_{-5} = w_{-4} = w_{-3} = w_{-2} = w_{-1} = w_0 = 1$. The initial values for $w_n = |I(P(n, M))|$ and $n \geq 1$ are

$$2, 3, 5, 7, 11, 15, 23, 32, 49, 69, 105, 149, \ldots.$$

Finally, we state the following corollary to Theorem 2 that can be proved in a standard way by the partial fraction expansion of the generating function $G(x)$ from Theorem 2.

Corollary 3. *Let M, V, $c(x)$, and $P(n, M)$ satisfy the conditions of Theorem 2. Also, ρ is the largest zero ($|\rho|$ is maximal among all the zeros) of the function*

$$Q(x) = (1-x)c(x) - x = 1 - x - x^2 + (1-x) \sum_{i=2}^{k} x^{a_i}.$$

Then asymptotically, the growth rate of $|I(P(n,M))|$ *is*

$$|I(P(n,M))| \lesssim c|\rho|^n$$

for some constant c.

References

[1] A. Burstein, S. Kitaev and T. Mansour, Independent sets in certain classes of (almost) regular graphs, preprint.

[2] N. Calkin and H. Wilf, The number of independent sets in a grid graph. *SIAM J. Discrete Math.* **11** (1998) 1, 54–60.

[3] R. Ehrenborg and S. Kitaev, Ferrers graphs, their independent sets and independence complexes, preprint.

[4] F. Forbes and B. Ycart, Counting stable sets on Cartesian products of graphs, *Discrete Math.* **186** (1998), 105–116.

[5] L. J. Guibas and A. M. Odlyzko, String overlaps, patterns matching, and nontransitive games, *Journal Comb. Theory Series A* **30** (1981), 19–42.

[6] M. J. Jou and G. J. Chang, Survey on counting maximal independent sets, *Proceedings of the Second Asian Mathematical Conference*, S. Tangmance and E. Schulz eds., World Scientific, Singapore (1995), 265–275.

[7] R. M. Karp, Reducibility amoung combinatorial problems, in: R. E. Miller, J. W. Thatcher (Eds.), Complexity of Computer Computations, Plenum Press, New York (1972), 85–103.

[8] T. Sillke, Counting independent sets, `http://www.mathematik.uni-bielefeld.de/~sillke/PROBLEMS/stable_sets`

[9] B. Winterfjord, Binary strings and substring avoidance, Master thesis, CTH and Göteborg University (2002).

E-mail address: `kitaev@ms.uky.edu`

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA