Segmented Partially Ordered Generalized Patterns

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Abstract

We continue the study of partially ordered generalized patterns (POGPs) considered in [9] for permutations and in [10] for words. We deal with segmented POGPs (SPOGPs). We state some general results and treat a number of patterns of length 4. We prove a result from [8] in a much simpler way and also establish a connection between SPOGPs and walks on lattice points starting from the origin and remaining in the positive quadrant. We give a combinatorial interpretation of the powers of the (generalized) Fibonacci numbers. The entire distribution of the maximum number of non-overlapping occurrences of a generalized pattern with no dashes in permutations or words studied in [9] and [10] respectively, has its counterpart in case of SPOGPs.

Keywords: pattern avoidance, segmented patterns, permutations, words, walks on lattice points, distribution

1 Introduction

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a *pattern* τ in a permutation π is "classically" defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31425 has three occurrences of the pattern 123, namely the subsequences 345, 145, and 125. Considering occurrences of patterns in permutations has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s.

In [1] Babson and Steingrímsson introduced *generalized permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563.

The motivation for introducing these patterns in [1] was the study of Mahonian statistics. A number of interesting results on generalized patterns were obtained in [5]. Relations to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there. However, the *segmented patterns* are of particular interest in this paper, which are the patterns with no dashes, that is, patterns correspond to contiguous subwords anywhere in a permutation (in some literature, we meet *subword patterns*, *contiguous patterns*, and *patterns with no gaps* meaning the same concept). Such patterns were considered, e.g., in [6] and [8].

In [9] introduced a further generalization of generalized patterns (GPs), namely *partially ordered generalized patterns* (*POGPs*). A POGP is a GP some of whose letters are incomparable. For instance, if we write $p = 1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25. A motivation for introducing POGPs is that they allow us to find the *exponential generating function e.g.f.* for the *entire distribution* of the maximum number of non-overlapping occurrences of a pattern p with no dashes, if we only know the e.g.f. $A(x)$ for the number of permutations that avoid p (we do know the e.g.f. for a lot of GPs with no dashes due to [6]). This distribution, according to [9, Th. 32], is given by

$$\frac{A(x)}{1-y-y(x-1)A(x)}.$$

In [10], the concept of POGPs in permutations is extended to that in words. Let $[k]^n$ denote the set of all the words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \ldots, k\}$. For example, if we write $\tau = 1\text{-}1'2'$, then we mean that in an occurrence of τ in a word $\sigma \in [k]^n$ the letter corresponding to the 1 in τ can be either larger, smaller, or equal to the letters corresponding to $1'$ and $2'$, whereas the letter corresponding to $1'$ must be less than the letter corresponding to $2'$. Thus, the word $113425 \in [5]^6$ contains seven occurrence of τ, namely 113, 134 twice, 125 twice, 325, and 425. Note, that a crucial difference with the case of permutations is that our patterns in words might have repeated letters. For instance, the pattern $\tau = 111'2'$ makes sense in the case of words, whereas there is no sense for τ in the case of permutations.

The main result in [10] is the *generating function* (*g.f.*) for the entire distribution of the maximum number of non-overlapping occurrences of a pattern τ with no dashes (and possibly with repeated letters) in k-ary words, provided we know the g.f. $A_\tau(x;k)$ for the number of k-ary words that avoid τ (we do know the g.f. for many of such patterns due to [2]–[4]). This distribution, according to [10], is given by

$$\frac{A_\tau(x;k)}{1-y-y(kx-1)A_\tau(x;k)}.$$

In the avoidance problems, any POGP may be viewed as a convenient short notation to a certain set of GPs. Indeed, to avoid a POGP is the same as to avoid simultaneously a number of certain GPs. For example, to avoid the POGP $21'1$ is the same as to avoid simultaneously the GPs 231, 312, and 321. From this point of view, study of POGPs is a continuation of study of multi-avoidance of usual GPs. In particular, results for the *segmented POGPS* (*SPOGPs*) considered in this paper are in the same direction as the results from [8] and some of the results from [6] and [9]. A POGP is segmented, if its occurrence in a permutation form a contiguous subword of the permutation of the corresponding length.

In section 2 we state two results (see proposition 1 and corollary 2) that are helpful when dealing with SPOGPs, and they are used in few places throughout the paper.

One SPOGP of length four, namely $122'1'$, was considered in [9, Sec. 6] (see proposition 3). In subsection 2.1 we study the number of permutations avoiding some other SPOGPs of length four. One of these POGPs, namely $12'21'$, give us a connection to walks on lattice points starting from the origin and remaining in the positive quadrant (see proposition 9 in subsection 2.2).

In [8], when finding the number of n-permutations avoiding simultaneously the GPs 123, 132, and 213, which is $A_n = \binom{n}{\lfloor n/2 \rfloor}$, rather complicated considerations were used. In section 3, we introduce *separated* SPOGPs (SSPOGPs), and give a "two line" proof of the result for A_n.

Although this paper is primarily concerned about occurrences of SPOGPs in permutations, in section 4 we extend a result on SSPOGPs for permutations to that for words. In particular, we give a combinatorial interpretation of the powers of the (generalized) Fibonacci numbers.

Finally, in section 5 we discuss the distribution of the maximum number of non-overlapping occurrences of SPOGPs in permutations and words. It turns out that the bivariate (exponential) generating functions for this distribution is given by the formulas stated above, and thus, once we solved the avoidance problem for a SPOGP, we are done with the maximum number of non-overlapping occurrences problem for this SPOGP. To explain this result, we introduce *multi-patterns* for SPOGPs which are a generalization of the multi-patterns considered in [9] for permutations and in [10] for words.

2 Segmented POGPs (SPOGPs)

Recall that a POGP is segmented, if its occurrence in a permutation form a contiguous subword of the permutation.

Let $Var(p)$ denote the alphabet of variables of the pattern p.

Let p be a SPOGP. A permutation π *quasi-avoids* p if π has exactly one occurrence of p and this occurrence consists of the $|p|$ rightmost letters of π. The concept of quasi-avoidance is helpful under certain enumeration problems (e.g., see the proof of corollary 2).

Proposition 1. *Let p be a SPOGP and $P(x)$ (resp. $P^*(x)$) be the e.g.f. for the number of permutations that avoid (resp. quasi-avoid) p. Then*

$$P^*(x) = (x-1)P(x) + 1.$$

Proof. One can copy the proof of [9, Prop. 4]. However, an alternative proof can be given. Let P_i (resp. P_i^*) be the number of i-permutations that avoid (resp. quasi-avoid) p. Also, suppose $|p| = k$, that is p consists of k letters. Then we first count the number of n-permutations containing p and subtract this number from $n!$ to get the desired.

Any permutation π containing p can be uniquely factored as $\pi = \pi_1\pi_2$, where π_1 quasi-avoids p and $|\pi_1| = i$. Clearly, $k \le i \le n$, and $P_j^* = 0$, for $1 \le j \le k-1$. Thus,

$$P_n = n! - \sum_{i=1}^{n} \binom{n}{i} P_i^*(n-i)!,$$

that is $P(x) = 1/(1-x) - P^*(x)/(1-x)$, which completes our proof. □

Corollary 2. *Let p be a SPOGP, the letter $a \notin Var(p)$, and a is incomparable to any $b \in Var(p)$. If $P(x)$ (resp. $P_a(x)$) is the e.g.f. for the number of permutations that avoid p (resp. the SPOGP pa) then $P_a(x) = xP(x) + 1$.*

Proof. The permutations avoiding pa are those avoiding p or quasi-avoiding p. Thus, according to proposition 1,

$$P_a(x) = P(x) + (x - 1)P(x) + 1 = xP(x) + 1.$$

□

2.1 Segmented POGPs of length four

The standard reverse and complement operations on permutations divide the patterns into equivalence classes. We may consider the following representatives of the equivalence classes of SPOGPs of length four : $12'21'$, $11'22'$, $122'1'$, $121'2'$, $11'2'2$, $12'1'2$, $1231'$, $1321'$, $2131'$, $121'3$, $131'2$, $231'1$, $1'1''12$, $1'11''2$, $1'121''$, and $11'1''2$. Note, that each of the three last patterns corresponds to simultaneous avoidance of two SPOGPs. For example, to avoid $1'1''12$ is the same as to avoid $1'2'12$ and $2'1'12$ simultaneously. Currently, we do not know solutions to the avoidance problem for the equivalence classes having the patterns $11'22'$, $121'2'$, $11'2'2$, $12'1'2$, $121'3$, $131'2$, and $231'1$. None of these classes is equivalent to another one. We record few initial values for the number of n-permutations avoiding these patterns, $n \geq 1$, in Table 1, and we leave consideration of these patterns as a challenging problem.

$11'22'$	1, 2, 6, 18, 70, 300, 1435, 7910, 47376, ...
$121'2'$	1, 2, 6, 18, 61, 281, 1541, 8920, 57924, ...
$11'2'2$	1, 2, 6, 18, 71, 322, 1665, 9789, 64327, ...
$12'1'2$	1, 2, 6, 18, 61, 272, 1410, 8048, 51550, ...
$121'3$	1, 2, 6, 20, 83, 411, 2290, 14588, 104448, ...
$131'2$	1, 2, 6, 20, 81, 390, 2161, 13678, 96983, ...
$231'1$	1, 2, 6, 20, 83, 402, 2245, 14192, 100650, ...

Table 1: The initial values for the number of n-permutations avoiding 4-SPOGPs in the unsolved cases, $n \geq 1$.

In all the propositions in this section, we assume B_n (resp. $B(x)$) denotes the number (resp. the e.g.f. for the number) of permutations that avoid a pattern under consideration.

The following result was obtained in [9].

Proposition 3. ([9, Th. 30]) *Let $p = 122'1'$. Then*

$$B(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^x \sin x) + \frac{1}{2}e^x \cos x.$$

We now consider some other patterns.

Proposition 4. *For the pattern $11'21''$ and $n \geq 1$, we have $B_n = n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}$.*

Proof. Clearly, to avoid $11'21''$ is the same as either to avoid the pattern $11'2$ or to quasi-avoid $11'2$. If A_n denotes the number of permutations that avoid

$11'2$ then according to the way the proof of [9, Prop. 4] goes, and to the considerations in the beginning of section 3, where we find A_n, we have

$$B_n = A_n + A^* = A_n + nA_{n-1} - A_n = n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

□

Proposition 5. *For the pattern* $1'1''12$ *and* $1'121''$*, we have* $B_0 = B_1 = 1$*, and, for* $n \geq 2$*,* $B_n = n(n-1)$*.*

Proof. If $\pi = \pi_1\pi_2 \cdots \pi_n$ avoids $1'1''12$ then there are no restrictions for $\pi_1\pi_2$ and $\pi_3\pi_4 \cdots \pi_n$ must be in decreasing order.

If $\pi = \pi_1\pi_2 \cdots \pi_n$ avoids $1'121''$ then there are no restrictions for π_1 and π_n, and $\pi_2\pi_3 \cdots \pi_{n-1}$ must be in decreasing order. □

Proposition 6. *For the pattern* $11'1''2$*, we have that*

$$B_n = \frac{n!}{\lfloor n/3 \rfloor! \lfloor (n+1)/3 \rfloor! \lfloor (n+2)/3 \rfloor!}.$$

Proof. As a corollary to theorem 11, with $\ell = 2$ and $A_n = 1$, we get that

$$B_n = \prod_{i=0}^{m_1-1} \binom{n - k_1 \cdot i}{k_1} \prod_{j=0}^{2-m_1} \binom{n - k_1 \cdot m_1 - k_2 \cdot j}{k_2},$$

where $k_1 = \lceil n/3 \rceil$, $k_2 = \lfloor n/3 \rfloor$, and $m_1 = n - 3 \cdot \lfloor n/3 \rfloor$. This expression for B_n can be seen to be equal to that we need to prove, by, e.g., checking the cases: $n = 3k$, $n = 3k+1$, and $n = 3k+2$. The initial values for B_n are 1, 2, 6, 12, 30, 90, 210, 560, 1680, 4200, ...

Another proof of the proposition is observing that the subsequences $\pi_1\pi_4\pi_7\cdots$, $\pi_2\pi_5\pi_8\cdots$, and $\pi_3\pi_6\pi_9\cdots$ must be in decreasing order. Thus we take all $n!$ permutations and divide them by the product of the numbers of permutations corresponding to the subsequences considered above. □

Proposition 7. *For the pattern* $1231'$*, we have that*

$$B(x) = xe^{x/2}\left(\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}x}{2}\right)^{-1} + 1;$$

and for the patterns $1321'$ *and* $2131'$*, we have that*

$$B(x) = x(1 - \int_0^x e^{-t^2/2}\ dt)^{-1} + 1.$$

Proof. According to proposition 2, our aim is to find the e.g.f.s for the number of permutations avoiding the segmented patterns 123, 132, and 213, which are given by [9, Th. 11, 12] and the discussions after those theorems in [9, Sec. 3]. □

Proposition 8. *For the pattern* $12'21'$*, we have that*

$$B_n = \binom{n-1}{\lfloor (n-1)/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor}.$$

Proof. Suppose an n-permutation π avoids $12'21'$.

If the letter 1 is in position t, that is $\pi(t) = 1$, then $\pi(t-2)\pi(t)$ (resp. $\pi(t)\pi(t+2)$) forms the pattern 21 (resp. 12). Since π avoids $12'21'$, $\pi(t-3)\pi(t-1)$ (resp. $\pi(t+1)\pi(t+3)$) must form the pattern 21 (resp. 12), which leads to $\pi(t-4)\pi(t-2)$ (resp. $\pi(t+2)\pi(t+4)$) must form the pattern 21 (resp. 12), and so on. This shows that the letters to the left (resp. to the right) of 1 having the same position parity as 1 does must decrease (resp. increase). The same holds for the letters to the left and to the right of 1 having the different position parity. There are no additional restrictions.

We distinguish two cases: n is even and n is odd. In each of these cases we consider two subcases: 1 is in an even position and it is in an odd position.

1) *n is even, $\pi^{-1}(1)$ is odd*: we choose the letters in the even positions in $\binom{n-1}{n/2}$ ways. Suppose now that 1 is in the $(i+1)$st odd position, where $0 \leq i \leq n/2-1$. We choose the letters in the odd positions to the left of 1 in $\binom{n/2-1}{i}$ ways, and we choose the letters in the even positions to the left of 1 in $\binom{n/2}{i}$ ways. Then we order all the letters uniquely according to the considerations above.

Similarly, we get the other cases.

2) *n is even, $\pi^{-1}(1)$ is even*: we have $\binom{n-1}{n/2}\sum_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i+1}$ permutations;

3) *n is odd, $\pi^{-1}(1)$ is odd*: we have $\binom{n-1}{(n-1)/2}\sum_{i=0}^{(n-1)/2}\binom{(n-1)/2}{i}\binom{(n-1)/2}{i}$ permutations;

4) *n is odd, $\pi^{-1}(1)$ is even*: we have $\binom{n-1}{(n-1)/2-1}\sum_{i=0}^{(n-1)/2-1}\binom{(n-1)/2-1}{i}\binom{(n+1)/2}{i+1}$ permutations.

Thus, B_n is given by

$$\begin{cases} \binom{n-1}{n/2}\left(\sum_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i} + \sum_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i+1}\right) & n \text{ is even,} \\ \binom{n-1}{(n-1)/2}\sum_{i=0}^{(n-1)/2}\binom{(n-1)/2}{i}^2 + \binom{n-1}{(n+1)/2}\sum_{i=0}^{(n-3)/2}\binom{(n-3)/2}{i}\binom{(n+1)/2}{i+1} & n \text{ is odd.} \end{cases}$$

To prove the statement, we need to prove that

$$\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{\lfloor\frac{n}{2}\rfloor} = \binom{n-1}{\lfloor\frac{n}{2}\rfloor}\sum_{i=0}^{\lceil\frac{n}{2}\rceil-1}\binom{\lceil\frac{n}{2}\rceil-1}{i}\binom{\lfloor\frac{n}{2}\rfloor}{i} + \binom{n-1}{\lceil\frac{n}{2}\rceil}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\binom{\lfloor\frac{n}{2}\rfloor-1}{i}\binom{\lceil\frac{n}{2}\rceil}{i+1},$$

where the righthand side is just rewritten expression for B_n.

If $n = 2k$ for some $k \geq 1$, then the equality to prove reduces to

$$\binom{2k}{k} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{k-i},$$

which is true using, e.g., a combinatorial proof with choosing a committee consisting of k people out of $k-1$ men and $k+1$ women.

If $n = 2k+1$ for some $k \geq 0$, then the equality to prove reduces to

$$\binom{2k}{k}\binom{2k+1}{k} = \binom{2k}{k}\sum_{i=0}^{k}\binom{k}{i}^2 + \binom{2k}{k+1}\sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1},$$

which holds since $\binom{2k}{k} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1} = \sum_{i=0}^{k}\binom{k}{i}\binom{k}{k-i}$, and $\binom{2k+1}{k} - \binom{2k}{k+1} = \binom{2k}{k}$. $\square$

2.2 Segmented POGPs and walks.

We refer to [11] for a survey on counting walks in sectors of the plane. Using a result by Guy [7] and proposition 8, we get the truth of the following proposition.

Proposition 9. *The number of $(n+1)$-permutations avoiding the SPOGP $12'21'$ is equal to the number of different walks of n steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the positive quadrant.*

Proof. We give a combinatorial proof of the statement by providing an explicit bijection between the $(n+1)$-permutations avoiding $12'21'$ and the walks of n steps in question. Clearly, any such walk can be coded by a word $w = w_1w_2\cdots w_n$ over the alphabet $\{a, \bar{a}, b, \bar{b}\}$ with the property that for any i, $1 \le i \le n$, the number of a's (resp. b's) in $w_1w_2\cdots w_i$ is not less than the number of $\bar{a}$'s (resp. $\bar{b}$'s) there. We may assume that a is the direction N, $\bar{a}$ – S, b – E, and $\bar{b}$ – W.

We start from an example and then make all required justifications. We use the structure of a permutation π avoiding $12'21'$, which according to the proof of proposition 8 is as follows: when reading π from left to right, the subsequence of π corresponding to the even positions, as well as the subsequence corresponding to the odd positions, to the left (resp. right) of the letter 1 are in decreasing (resp. increasing) order.

Suppose we are given a walk $w = ba\bar{a}b\bar{b}aa\bar{b}$. The i-th letter in w will correspond to the letter $(i+1)$ in the $(n+1)$-permutation π corresponding to w. If the i-th letter in w is a or $\bar{a}$, then the letter $(i+1)$ is to the left of the letter 1 in π. Otherwise, $(i+1)$ is to the right of 1 in π. We arrange the letters to the left (resp. right) of 1 in decreasing (resp. increasing) order, and keeping the bar if any, get the first approximation of π: $87\bar{4}3125\bar{6}\bar{9}$. Now, the bared letters will move toward 1 jumping over the neighbor with no bar; if more than one of consecutive letters are bared, these bared consecutive letters form a group that will jump toward 1 over the closest neighbor with no bar. After a jump, we erase bar from the largest letter in a group of letters that have jumped together to get the second approximation of π: $873412\bar{6}95$. In general, only one group of bared letters jump in a time in each of two parts (to the right and to the left of 1), namely the most far group from 1. Under this procedure a merging of two bared groups is possible. We are supposed to proceed until all the letters have no bar. In our example one more jump is required, and we get $\pi = 873416295$. One can easily see that π avoids $12'21'$ because of its structure. At this point we make a general observation, that no matter how many jumps are required, the letters to the right (resp. left) of 1 will keep being to the right (resp. left) according to the properties of w (we deal with a walk in the first quadrant). So, the operation of a jump is well-defined.

Before describing a reverse to the procedure above and proving that we have a bijection (in general case), we make the following observation that simplifies our exposition. For any walk w, a's and $\bar{a}$'s have no affect to b's and $\bar{b}$'s and thus we can treat them separately, which corresponds to treating separately the letters of π to the right and to the left of 1. Thus, we can start from considering only the letters of the first approximation of π (some of them might be bared) corresponding to the positions of, say, b's and $\bar{b}$'s, and we write them in increasing order. Clearly, we can use the concept of order-isomorphism and reduce the initial problem to the following: "Given a word $u = 12\cdots n$ some of whose letters might be bared in a way that u corresponds to a walk on the positive x-axis after associating its letters with b's and $\bar{b}$'s depending on whether a letter is bared or not. Find a bijection between the set of all such words and all the words over $\{1, 2, \dots, n\}$ having the property that its letters in the even positions, as well as the letters in the odd positions, when reading from left to right are in increasing order." Clearly, when talking about a's and $\bar{a}$'s instead of b's and $\bar{b}$'s we change the word "increasing" to "decreasing" in the auxiliary problem above, which from now on will be the main problem under consideration.

To get a word v from a bared word u, we proceed using the procedure of consecutive jumps to the left described in the example above. This is a well-defined operation as it is mentioned above, and all the bars will disappear after several jumps. However, we need to make sure that the obtained word v has the good properties, that is, it is obtained by shuffling two increasing sequences. We only need to consider the following (we assume that $i < j$ everywhere below). For any two letters with no bars i and j under the jumping procedure, i is always to the left of j, which cannot course any problem. On the other hand, if i and j are bared then, even taking into account possible merging of bared blocks, i will always be to the left of j – this is good . Finally, we need to check what happens if when jumping, a letter i with no bar is to the right of j after the procedure is done (the case i with a bar and j with no bar is not needed to be considered since our jumps are to the left). To this end, let B denote the subword corresponding to the bared block in u containing j. As B starts moving toward 1 (possibly after merging with another bared block, that is, in fact, j starts moving with the bared block BB', where B' might be empty), we get a sequence of bared blocks obtained from BB' after jumping, which eventually disappears, that is, we have

$$BB' = B_0 \to B_1 \to \cdots \to B_k = \{\bar{t}\} \to \epsilon,$$

where t is a letter, and ϵ is the empty word. The main observation is that all the letters of BB' will have the same position parity, whereas all the letters to be jumped over will have the different position parity (that is easy to see). Thus, since we jump over i at some point, i and j will have different position parity and this is ok for i to be to the right of j at the end of the jumping procedure (the relative order of different parity letters is irrelevant).

To see that the jumping procedure is injective, assume that we have two different bared words u_1 and u_2. Moreover, let i be the rightmost position such that i is bared in u_1 and i is not bared in u_2 or vice versa. Such an i obviously exists, and $(i+1)(i+2)\cdots n$ is with no bars in the both words. After implementing our jumps, the letter i with no bar stays at its original place (as the letters $(i+1), (i+2), \dots, n$ do), whereas $\bar{i}$ moves to the left, thus we get different words as the outcomes.

We explain a reverse to the jumping procedure for the auxiliary problem, but it will be clear how to extend it to the case of our original problem, and we will give an example for the initial problem after our explanation.

We are given a word v obtained by shuffling two increasing sequences; our purpose is to get the bared word u corresponding to v. We rewrite v going from left to right, position by position, as follows: if at step i, either i or $\bar{i}$ is in position i, we proceed to the next step. If at step i the letter i or $\bar{i}$ is at position j, where $i < j$ (the case $j < i$ will not be possible according to the procedure we are describing), and $w = u_1 u_2 \cdots u_{i-1} w_i w_{i+1} \cdots w_n$ is the current approximation of u (note that we have already obtained the first $(i-1)$ letters of u, and $w_j = i$ or $w_j = \bar{i}$), then we move w_j into position i and make the block

$w_i w_{i+1} \cdots w_{j-1}$ to be bared (here we assume that $\bar{\bar{e}} = \bar{e}$ for any letter e). This reverse procedure requires $n-1$ steps. To see that we get a bared word with the right properties, namely that at each step when going from left to right the number of bared letters does not exceed the number of letters with no bars, we may proceed by induction on i, where i corresponds to the i-th step.

If $i = 1$ then either the letter 1 is in the first position from the beginning, or it comes there from the second position. In either case 1 has no bar, and we can possibly create at most one bar, which makes the first approximation of u good.

Suppose now that the statement is true for $i-1$, that is, the $(i-1)$st approximation of u has the right properties. As above, assume that after $(i-1)$ steps, we have $w = u_1 u_2 \cdots u_{i-1} w_i w_{i+1} \cdots w_n$, where $w_j = i$ for some $j > i$. The crucial observation is that the letters of $w_i w_{i+1} \cdots w_{j-1}$ must be either all bared, or all with no bars, or the bared letters are followed by the letters with no bars when reading from the left. No mixing of bared and unbared letters is possible (indeed, a bar appears when one of the letters u_1, u_2, $\ldots, u_{i-1}$ is moving to its terminate place, and all the letters that are jumped over must receive a bar). On the other hand, if, say, $w_k w_{k+1} \cdots w_{j-1}$ is with no bars for some $k \geq i$, then $w_k w_{k+1} \cdots w_{j-1} i$ must be a subword of the initial word v. In particular, it must be that $w_{j-2} < i$, which is contradiction unless $k = j-1$ or $k = j$ (indeed, all the letters $1, 2, \ldots, (i-1)$ are in the beginning of w). If $k = j-1$, since there is no mixing of bars, i is unbared, and moving it to the terminate position creates one extra bar, which is compensated by i and we are fine by induction. If $k = j$, no extra bar will be created, and it does not matter if i is bared or unbared, we are fine by induction.

We end up the proof of the proposition by an example of implementing our reverse operation for the initial problem. We simply implement the operation described above separately for the letters staying to the right, respectively to the left, of 1.

$$483516297 \to 48\bar{5}312\bar{6}97 \to \bar{8}\bar{5}4312\bar{6}7\bar{9} \to baa\bar{a}\bar{b}b\bar{a}\bar{b}.$$

□

3 Separated segmented POGPs (SSPOGPs)

Definition 10. Suppose $p = a_1 a_2 \cdots a_k$ is a permutation and, for fixed non-negative integers $\ell_1, \ell_2, \ldots, \ell_{k-1}$, the letters $b^{(i,j)}$, $1 \leq i \leq k-1$, $1 \leq j = j(i) \leq \ell_i$, are incomparable neither with each other nor with a_is, $1 \leq i \leq k$. We call the SPOGP

$$a_1 b^{(1,1)} b^{(1,2)} \cdots b^{(1,\ell_1)} a_2 b^{(2,1)} b^{(2,2)} \cdots b^{(2,\ell_2)} a_3 \cdots a_{k-1} b^{(k-1,1)} b^{(k-1,2)} \cdots b^{(k-1,\ell_{k-1})} a_k$$

separated segmented POGP (*SSPOGP*). For the SSPOGP above we use the notation

$$\tau_k(\ell_1, \ell_2, \ldots, \ell_{k-1}) = a_1|_{\ell_1} a_2|_{\ell_2} a_3 \cdots a_{k-1}|_{\ell_{k-1}} a_k.$$

We use "$|$" instead of "$|_1$".

The idea of considering patterns in permutations having the properties like the pattern $\tau_k(\ell_1, \ell_2, \ldots, \ell_{k-1})$ has, that is when in any occurrence of a pattern in permutations, the distance between any two letters from this occurrence is a fixed positive number, does not seem to be new. However, it might be useful to realize that such patterns form a subclass of POGPs.

We have already met SSPOGPs in subsection 2.1, e.g., $p = 121'3$ (here $\ell_1 = 0$ and $\ell_2 = 1$). However, the easiest example of a SSPOGP is the pattern $11'2 = 1|2$. There are $\binom{n}{\lfloor n/2 \rfloor}$ permutations avoiding this pattern. Indeed, we choose the letters of our permutation in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too. We used the property that the letters in odd and even positions do not affect each others.

A rather simple argument above, when considering the pattern $1|2$, considerably simplifies the proof of [8, Th. 1] after observing that to avoid $1|2$ is the same as to avoid simultaneously the SPOGPs 123, 132, and 213. The simple idea of separating a permutation into parts that do not affect each others gives the number of permutations that avoid a SSPOGP having $\ell_i = \ell_j$ for all i, j. We record this result into the following theorem which is easy to prove using the observation on separation.

Theorem 11. *Let A_n denote the number of n-permutations avoiding the segmented pattern $a_1a_2 \cdots a_k$; $k_1 = \lceil n/(\ell+1) \rceil$, $k_2 = \lfloor n/(\ell+1) \rfloor$, $m_1 = n - (\ell + 1)\lfloor n/(\ell+1) \rfloor$, and $m_2 = (\ell+1)(\lfloor n/(\ell+1) \rfloor + 1) - n$. Then, for $\ell \geq 0$, the number B_n of n-permutations avoiding the SSPOGP $\tau_{k,\ell} = a_1|_\ell a_2|_\ell a_3 \cdots a_{k-1}|_\ell a_k$ is given by*

$$A_{k_1}^{m_1} A_{k_2}^{m_2} \prod_{i=0}^{m_1-1} \binom{n - k_1 \cdot i}{k_1} \prod_{j=0}^{m_2-1} \binom{n - k_1 \cdot m_1 - k_2 \cdot j}{k_2}.$$

Proof. When using the idea of separation, clearly there are m_1 subsequences of length k_1 and m_2 subsequences of length k_2. The rest is easy to see. □

Remark 12. If n is devisible by $\ell+1$, the result of theorem 11 can be simplified as

$$B_n = A_{n/(\ell+1)}^{\ell+1} \prod_{i=0}^{\ell} \binom{n(\ell - i + 1)/(\ell + 1)}{n/(\ell + 1)}.$$

In particular, if $\ell = 0$ then $B_n = A_n$.

Remark 13. We can use the result in theorem 11 to get an explicit number of permutations avoiding $\tau_{k,\ell}$ in a lot of particular cases due to the results in [6, 9] (we simply use explicit values of A_n whenever we know them).

4 Separated segmented POGPs and words

Similarly to definition 10 one can define a separated segmented POGP when considering words rather than permutations. We may also allow $a_1a_2 \cdots a_k$ from definition 10 to be a word.

An analogue of theorem 11 is the following theorem.

Theorem 14. *Let $A(n;p)$ denote the number of all words from $[p]^n$ avoiding the segmented pattern $a_1a_2 \cdots a_k$ over $[m]$; $k_1 = \lceil n/(\ell+1) \rceil$, $k_2 = \lfloor n/(\ell+1) \rfloor$, $m_1 = n - (\ell+1)\lfloor n/(\ell+1) \rfloor$, and $m_2 = (\ell+1)(\lfloor n/(\ell+1) \rfloor + 1) - n$. Then, for $\ell \geq 0$, the number $B(n;p)$ of words from $[p]^n$ that avoid the SSPOGP $\tau_{k,\ell,m} = a_1|_\ell a_2|_\ell a_3 \cdots a_{k-1}|_\ell a_k$ is given by $A^{m_1}(k_1;p)A^{m_2}(k_2;p)$.*

Remark 15. We can use the result in theorem 14 to get explicit numbers of words over a p-letter alphabet avoiding $\tau_{k,\ell,m}$ in many particular cases due to the results in [2]–[4] (we can use explicit values of $A(n;p)$ whenever we know them).

This is well known and it is not difficult to see that the number of different binary strings of length n that avoid the segmented pattern 11 is given by F_{n+2}, where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. This fact alone with theorem 14 give us a combinatorial interpretation of the powers of the Fibonacci numbers. Indeed, for $n \geq 2$ and $\ell \geq 0$, the number of binary words of length $(\ell+1)\cdot(n-2)$ avoiding the SSPOGP $1|_\ell 1$ is given by $F_n^{\ell+1}$. Likewise, the number of binary words of certain lengths avoiding a SSPOGP pattern $1|_\ell 1 \cdots 1|_\ell 1$ is given by a certain power of a *generalized Fibonacci number*.

5 The distribution of non-overlapping SPOGPs

We define a *multi-pattern* to be a pattern $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_t$, where $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ is a set of SPOGPs and each letter of σ_i is incomparable with any letter of σ_j whenever $i \neq j$. Our multi-patterns are a generalization of the multi-patterns considered in [9] for permutations and in [10] for words.

It turns out that we can copy all the arguments from [9] and [10], when considering the multi-patterns there, to get results for our multi-patterns. Indeed, the fact that in a pattern σ_i, for some i, some of the letters might be incomparable does not affect any of the considerations. In particular, [9, Th. 28] (resp. [10, Th. 4.1]) gives a way to obtain the e.g.f. (resp. g.f.) for the number of permutations (resp. words) avoiding a multi-pattern $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_t$, provided we know the e.g.f. (resp. g.f.) for the number of permutations (resp. words) that avoid the SPOGP σ_i for each i, $1 \leq i \leq t$.

Theorem 16 gives an interesting application of the multi-patterns in finding a certain statistic, namely the *maximum number of non-overlapping occurrences of a SPOGP* in permutations and words. For instance, the maximum number of non-overlapping occurrences of the SPOGP $11'2$ in the permutation 621394785 is 2, which is given by the occurrences 213 and 478, or the occurrences 139 and 478.

Theorem 16 can be proven in the same way its counterparts [9, Th. 32] and [10, Th. 5.1] were proven.

Theorem 16. *Let p be a SPOGP. Let $B(x)$ (resp. $B(x;k)$) be the e.g.f. (resp. g.f.) for the number of permutations (resp. words over $[k]$) that avoid p. Let $D(x,y) = \sum_\pi y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ and $D(x,y;k) = \sum_{n\geq 0}\sum_{w\in[k]^n} y^{N(w)}x^n$ where $N(s)$ is the maximum number of non-overlapping occurrences of p in s. Then*

$$D(x,y) = \frac{B(x)}{1 - y(1 + (x-1)B(x))}$$

and

$$D(x,y;k) = \frac{B(x;k)}{1 - y(1 + (kx-1)B(x;k))}.$$

The following examples are corollaries to Theorem 16.

Example 1. If we consider the SPOGP $11'$ then clearly $B(x) = 1 + x$ and $B(x;k) = 1 + kx$. Hence,

$$D(x,y) = \frac{1+x}{1-yx^2} = \sum_{i\geq 0}(x^{2i} + x^{2i+1})y^i,$$

and

$$D(x,y;k)=\frac{1+kx}{1-y(kx)^2}=\sum_{i\geq 0}((kx)^{2i}+(kx)^{2i+1})y^i,$$

which is easy to see to be true.

Example 2. If we consider permutations and the SPOGP $122'1'$ then $B(x)$ is given by proposition 3, and the distribution of the maximum number of non-overlapping occurrences of $122'1'$ is given by the formula

$$D(x,y)=\frac{\frac{1}{2}+\frac{1}{4}\tan x(1+e^{2x}+2e^x\sin x)+\frac{1}{2}e^x\cos x}{1-y(1+(x-1)(\frac{1}{2}+\frac{1}{4}\tan x(1+e^{2x}+2e^x\sin x)+\frac{1}{2}e^x\cos x))}.$$

If we are interested in, say, just one non-overlapping occurrence of $122'1'$, we consider the coefficient of y in the expansion of $D(x,y)$:

$$\frac{1}{4}x^4+\frac{9}{20}x^5+\frac{13}{20}x^6+\frac{23}{30}x^7+\frac{143}{180}x^8+\frac{301}{405}x^9+\frac{2591}{4050}x^{10}+\cdots.$$

That is, the initial values for the number of permutations having one non-overlapping occurrence of $122'1'$ are 0, 0, 0, 0, 6, 54, 468, 3864, 32032, 269696, 2321536, ….

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