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Asymptotically Optimal Lower Bounds For the Condition Number of a Real Vandermonde Matrix[1]

Ren-Cang Li[2]

June 2004
Revised July 2004

ABSTRACT

Lower bounds on the condition number $\min \kappa_p(V)$ of a Vandermonde matrix V are established in terms of the dimension n or n and the largest absolute value among all nodes that define the Vandermonde matrix. Many bounds here are asymptotically sharp, and compare favorably to those of the same kind due to Gautschi and Inglese (*Numer. Math.*, 52 (1988), 241–250) who considered either all positive nodes or real symmetric nodes and those due to Tyrtyshnikov (*Numer. Math.*, 67 (1994), 261–269). As consequences our bounds imply that $\min \kappa_p(V)$ over all possible real nodes goes to ∞ as fast as $(1+\sqrt{2}\,)^n$, and $\min \kappa_p(V)$ over all possible nonnegative nodes goes to ∞ as fast as $(1+\sqrt{2}\,)^{2n}$. Extensions were made to confluent Vandermonde matrices and rectangular Vandermonde matrices, including brief outlines on how to deal with Vandermonde-like matrices and complex Vandermonde matrices.

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[1]This report is available on the web at `http://www.ms.uky.edu/~math/MAreport/PDF/04-05.pdf`.

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Lower bounds on the condition number $\min \kappa_p(V)$ of a Vandermonde matrix V are established in terms of the dimension n or n and the largest absolute value among all nodes that define the Vandermonde matrix. Many bounds here are asymptotically sharp, and compare favorably to those of the same kind due to Gautschi and Inglese (*Numer. Math.*, 52 (1988), 241–250) who considered either all positive nodes or real symmetric nodes and those due to Tyrtyshnikov (*Numer. Math.*, 67 (1994), 261–269). As consequences our bounds imply that $\min \kappa_p(V)$ over all possible real nodes goes to ∞ as fast as $(1+\sqrt{2}\,)^n$, and $\min \kappa_p(V)$ over all possible nonnegative nodes goes to ∞ as fast as $(1+\sqrt{2}\,)^{2n}$. Extensions were made to confluent Vandermonde matrices and rectangular Vandermonde matrices, including brief outlines on how to deal with Vandermonde-like matrices and complex Vandermonde matrices.

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Contents

1 Introduction

Given n numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ called *nodes*, the associated *Vandermonde Matrix* is defined as

$$V \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}. \tag{1.1}$$

It is perhaps one of the best known structural matrices, arising from polynomial interpolation and others [3]. It is invertible if all nodes α_j are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, and in fact

$$\det(V) = \prod_{n \geq i > j \geq 1} (\alpha_i - \alpha_j).$$

Vandermonde matrices are notoriously ill-conditioned [14, p.428], i.e., its condition number can become arbitrarily large, even for modest n. This is not surprising because moving one node arbitrarily close to another will make V arbitrarily close to a singular matrix. Therefore the question of importance about V is not *how bad a Vandermonde matrix V can be* but rather *what one can hope for at best from V* as far as its condition number is concerned. In the past, Gautschi and his coauthor had systematically studied the condition number estimation in [7, 8, 9, 10, 12], where various condition number bounds in terms of the nodes α_j have been established, as well as bounds in terms of dimension n only. Most notably, in [12] two lower bounds in terms of n were obtained for positive nodes ($\alpha_j \geq 0$) and real symmetric nodes ($\alpha_j + \alpha_{n+1-j} = 0$) to which we shall return for comparison purposes later. A similar bound was obtained by Tyrtyshnikov [20], too. It is worth mentioning that despite of its notoriously ill-conditioning, there is a way to compute its singular value decompositions to a highly relative accuracy [6, 16].

Although V is well-defined no matter if all or some of α_j are real or complex, this paper is confined to real Vandermonde matrix V only, i.e., *all α_j are real*, except briefly in Section 12. Throughout this paper, some notation is *exclusively* reserved for one assignment, including V and its nodes α_j and $\alpha \stackrel{\text{def}}{=} \max_j |\alpha_j|$, along with many others in Table 1.1. V_{sym} is one of those V whose nodes are real symmetric.

The major objective of this paper is to bound the ℓ_p-condition number $\kappa_p(V) = \|V\|_p \|V^{-1}\|_p$ from below, in terms of n or n and α. Many asymptotically optimal bounds have been established. By *asymptotically optimal bounds* we mean those that will give

$$\text{asymptotic speed} \stackrel{\text{def}}{=} \lim_{n \to \infty} \left[\min_{\cdots} \kappa_p(V) \right]^{1/n} \tag{1.2}$$

exactly, where $\min\limits_{\cdots}$ is taken over some prescribed subset or the entire set of Vandermonde matrices. Particular attention will be given to the case $p = \infty$. In a sense, considering $p = \infty$ is sufficient because of the exponential growth of $\kappa_\infty(V)$ and because (see (3.9))

$$n^{-2/p} \kappa_p(V) \leq \kappa_\infty(V) \leq n^{2/p} \kappa_p(V), \tag{1.3}$$

and thus they all have the same asymptotic speed. Nonetheless, whenever it is possible to establish sharper bounds on $\kappa_p(V)$ *directly* instead of *indirectly* through bounds on $\kappa_\infty(V)$ combined with (1.3), we shall go for the sharper ones. To show what kind of consequences our bounds may imply, we give these examples:

$$\lim_{n\to\infty}\left[\min_{\alpha_j}\kappa_p(V_{\rm sym})\right]^{1/n}=\lim_{n\to\infty}\left[\min_{\alpha_j}\kappa_p(V)\right]^{1/n} = 1+\sqrt{2}, \tag{1.4}$$

$$\lim_{n\to\infty}\left[\min_{\alpha_j\ge 0}\kappa_p(V)\right]^{1/n} = (1+\sqrt{2}\,)^2. \tag{1.5}$$

What we can draw from them is that for best possibly conditioned V and $V_{\rm sym}$, without restriction on nodes α_j, both $\kappa_p(V)$ and $\kappa_p(V_{\rm sym})$ go to ∞ as fast as $(1+\sqrt{2}\,)^n$, and that subject to all α_j being nonnegative $\kappa_p(V)$ goes to ∞ as fast as $(1+\sqrt{2}\,)^{2n}$. Related, roughly speaking it was proved in Gautschi and Ingese [12] that subject to all α_j being nonnegative the best possible condition number goes to ∞ *at least* as fast as 2^n, while with symmetric nodes it goes to ∞ *at least* as fast as $(\sqrt{2}\,)^n$. In Tyrtyshnikov [20], this is improved to the point that the best possible condition number goes to ∞ *at least* as fast as 2^n, regardless of the nodes being nonnegative or symmetrical. Clearly they are far from asymptotically optimal.

Asymptotically optimal results subject to α being fixed or bounded, are obtained here, too. They seem to be the first of their kind. For example, we shall have

$$\lim_{n\to\infty}\left[\min_{\alpha\le 1/2}\kappa_p(V)\right]^{1/n}=\lim_{n\to\infty}\left[\min_{\alpha=1/2}\kappa_p(V)\right]^{1/n}=2+\sqrt{5}$$

which remains true with V replaced by $V_{\rm sym}$.

Another instructive outcome of our study shows that both $\min_{\alpha_j}\kappa_p(V)$ and $\min_{\alpha_j\ge 0}\kappa_p(V)$ subject to α being fixed behave qualitatively, as functions of α, like Figure 1.1, where initially as α increases, both $\min_{\alpha_j}\kappa_p(V)$ and $\min_{\alpha_j\ge 0}\kappa_p(V)$ decrease until at $\alpha=\alpha_{\rm opt}$ when global minimums of $\kappa_p(V)$ are reached, and then they start climbing again. Notice $\alpha_{\rm opt}$ may be different for the two, but $\alpha_{\rm opt}=\mathcal{O}(1)$ in both cases.

Although our study here does not yield optimally conditioned V, i.e., V that achieves $\min\kappa_p(V)$ under various circumstances, it does, however, say that a nearly optimally conditioned V with nodes in $[-\alpha,\alpha]$ (i.e., with α fixed, α_j's vary within the interval) is the one defined with the translated Chebyshev nodes in a slightly larger interval (so that the two exterior nodes are $-\alpha$ and α, respectively) and similarly for V with nodes in $[a,b]$, where $0\le a<b=\alpha$. If all α_j are allowed to vary freely along the entire real line, a nearly optimally conditioned V is the one defined with Chebyshev nodes (for which $\alpha=\cos\frac{\pi}{2n}\approx 1$); If all α_j are forced nonnegative but otherwise free, a nearly optimally conditioned V is the one defined with the translated Chebyshev nodes in the interval $[a,b]=[0,1]$. However, those nearly optimally conditioned V are truely by the word "nearly", that is to say they are just nearly optimal but not optimal, according to those few optimally conditioned V computed by [9] under the condition that the optimal V is unique (for any fixed n).

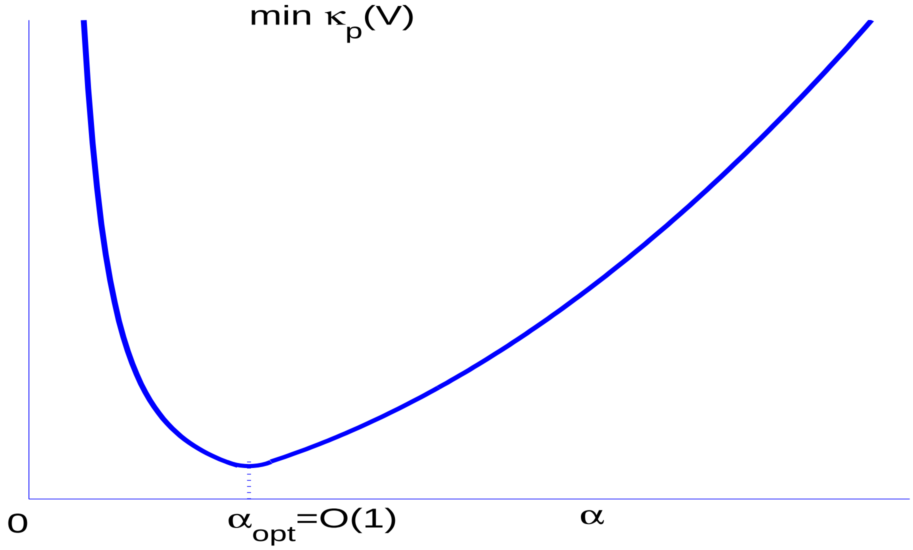


Figure 1.1: Qualitative behaviors of $\min_{\alpha_j} \kappa_p(V)$ and $\min_{\alpha_j \geq 0} \kappa_p(V)$ as α varies

We also extend our results for V to Confluent Vandermonde matrices $V_{\rm c}$ and rectangular Vandermonde matrices, i.e., the sub-matrix V_k of the first k rows of V, and will briefly explain how to apply our main technique to Vandermonde-like and complex Vandermonde matrices.

The rest of this paper is organized as follows. A cornerstone of our study is the use of the absolute sums of the coefficients of the translated Chebyshev polynomials of the first kind. They are defined and computed for a symmetric interval or a nonnegative interval in Section 2. Section 3 reviews the ℓ_p-vector and ℓ_p-operator norms, and norm equivalence relations, and in particular we argue that the asymptotical speed of $\kappa_p(V)$ is independent p. Our upper bounds on $\min \kappa_p(V)$ are obtained by the computations for V with the translated Chebyshev nodes. This is done in Section 4. Section 5 proves a general bound on V with nodes restricted to a given interval $[a, b]$. Section 6 applies the general bound to derive various asymptotically optimal bounds on $\min \kappa_p(V)$ with or without fixing α. Equation (1.4) is a consequence of results in the section. Section 7 applies the general bound in Section 5 to the case when all $\alpha_j \geq 0$ and derives various asymptotically optimal bounds, including those that lead to (1.5). Sections 8, 9, and 10 consider rectangular Vandermonde matrices, where results are very much parallel to those in Sections 5, 6, and 7. Attempts are made in Section 11 to extend those results to a confluent Vandermonde matrix $V_{\rm c}$; however, we only succeed in establishing lower bounds on $\kappa_p(V_{\rm c})$ without knowing whether these bounds are asymptotically optimal. Section 12 explain how to apply our main technique to Vandermonde-like and complex Vandermonde matrices. Finally Section 13 draws a few concluding remarks, presents open problems, and outlines some future work.

Notation We shall stick to the global assignments in Table 1.1, unless otherwise explicitly stated. $1 \leq p \leq +\infty$ and $1/p + 1/p' = 1$. Besides those, $\mathbb{R}^{m\times n}$ is the set of all $m \times n$

V, α_j, α	Vandermonde matrix V, its n nodes, and $\alpha = \max_j \lvert\alpha_j\rvert$;
$V_{\rm sym}$	V with symmetric nodes: $\alpha_j + \alpha_{n+1-j} = 0$;
$[a, b]$	the interval that contains all nodes α_j, see (5.1);
ω, τ	parameters defining transformation $t(x) = x/\omega + \tau$, and whenever there is $[a, b]$ in the context, they are defined by (4.3);
$\mathbb{V}_{[\gamma,\delta]}$	$\{V \,:\, \text{all } \alpha_j \in [\gamma, \delta]\}$;
$T_n(t)$, $T_n(x;\omega,\tau)$	Chebyshev polynomial, its translation $T_n(x;\omega,\tau) \stackrel{\rm def}{=} T_n(x/\omega + \tau)$;
θ_j, t_j	$\theta_j = \frac{2j-1}{2n}\pi$, and $t_j = \cos\theta_j$: zeros of $T_n(t)$, defined by (4.1);
x_j	$x_j = \omega(t_j - \tau)$: zeros of $T_n(x;\omega,\tau)$, defined by (4.2);
$a_{jn} \equiv a_{jn}(\omega,\tau)$	coefficients of $T_n(x;\omega,\tau)$ defined by (2.6);
$S_{n,p}(\omega,\tau)$	$\left(\sum_{j=0}^{n} \lvert a_{jn}\rvert^p\right)^{1/p}$ defined by (2.8).

Table 1.1: Special Notation

real matrices, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ and $\mathbb{R} = \mathbb{R}^1$. $\mathsf{sign}(\xi)$ is the sign of $\xi \in \mathbb{R}$. $\lfloor\xi\rfloor$ is the largest integer that is smaller than ξ; while $\lceil\xi\rceil$ is the smallest integer that is larger than ξ. For two sequences of numbers: a_n and b_n,

$a_n \sim b_n$	means $a_n/b_n \to 1$	as $n \to +\infty$;
$a_n = \mathcal{O}(b_n)$	means $c_1 \le a_n/b_n \le c_2$	for constants c_1 and c_2 and n large enough;
$a_n = \mathcal{O}_n(b_n)$	means $n^{d_1} \le a_n/b_n \le n^{d_2}$	for constants d_1 and d_2 and n large enough.

In our use later in this paper, both a_n and b_n grow exponentially in n, and thus the hidden factors n^{d_i} in $a_n = \mathcal{O}_n(b_n)$ are less significant, comparing to the exponential growth. It can be proved that

$$a_n = \mathcal{O}_n(b_n) \quad \Rightarrow \quad \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} b_n^{1/n} \tag{1.6}$$

if one of the limits exists.

2 Coefficients of Chebyshev polynomials

Chebyshev polynomials of the first kind can be defined recursively as

$$\begin{aligned} T_0(t) &= 1, \\ T_1(t) &= t, \\ T_n(t) &= 2t\,T_{n-1}(t) - T_{n-2}(t) \quad \text{for } n \ge 2. \end{aligned}$$

$T_n(t)$ is the nth Chebyshev polynomial. Alternatively on the interval $[-1, 1]$, $T_n(t) = \cos(n \arccos t)$, and thus

$$T_n(-t) = \cos(n(\pi - \arccos t)) = (-1)^n T_n(t). \tag{2.1}$$

Other useful formulas are

$$T_n(t) \;=\; \frac{1}{2}\left(t + \sqrt{t^2-1}\right)^n + \frac{1}{2}\left(t - \sqrt{t^2-1}\right)^n, \tag{2.2}$$

$$= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} t^{n-2j} \left(t^2 - 1\right)^j. \tag{2.3}$$

The nth *Translated Chebyshev Polynomial* is defined by

$$T_n(x; \omega, \tau) \stackrel{\text{def}}{=} T_n(x/\omega + \tau), \tag{2.4}$$

obtained upon applying the transformation

$$t(x) = x/\omega + \tau. \tag{2.5}$$

Here and in the rest of this paper T_n is overloaded with distinctions according to its argument(s). It can be seen that $T_n(x; \omega, \tau)$ is a polynomial of degree n in x. Write

$$T_n(x; \omega, \tau) = a_{nn} x^n + a_{n-1\,n} x^{n-1} + \cdots + a_{1n} x + a_{0n}, \tag{2.6}$$

where $a_{jn} \equiv a_{jn}(\omega, \tau)$ are functions of ω and τ which, wherever referenced, should be either clear from the context or will be explicitly stated. It can be seen that

$$a_{nn} = 2^{n-1}/\omega^n \quad \text{for } n \geq 1. \tag{2.7}$$

But closed formulas for a_{jn} for all $j < n$ are hard to get. Luckily we do not need them. What is needed is

$$S_{n,p}(\omega, \tau) \stackrel{\text{def}}{=} \left(\sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}, \tag{2.8}$$

a function of ω and τ, too. Successful computation of $S_{n,p}(\omega, \tau)$ is crucial to our later development. But in its generality, an explicit formula for $S_{n,p}(\omega, \tau)$ is hard to come by, too. Nevertheless we still manage to find formulas for $S_{n,1}(\omega, \tau)$ for two different cases $\tau = 0$ or $|\tau| \geq 1$. Notice $S_{n,p}(\omega, \tau)$ and $S_{n,1}(\omega, \tau)$ are related by the following inequalities

$$(n+1)^{-1/p'} S_{n,1}(\omega, \tau) \leq S_{n,p}(\omega, \tau) \leq S_{n,1}(\omega, \tau). \tag{2.9}$$

The first inequality can be improved when $\tau = 0$ for which, by Theorem 2.2, only $\lceil (n+1)/2 \rceil$ of all a_{jn} are nonzero. Therefore,

$$\lceil (n+1)/2 \rceil^{-1/p'} S_{n,1}(\omega, 0) \leq S_{n,p}(\omega, 0) \leq S_{n,1}(\omega, 0). \tag{2.10}$$

Both (2.9) and (2.10) can be proved by using Hölder inequality

$$\sum_{j=1}^{m} |\xi_j \zeta_j| \leq \left(\sum_{j=1}^{m} |\xi_j|^p \right)^{1/p} \left(\sum_{j=1}^{m} |\zeta_j|^{p'} \right)^{1/p'} \tag{2.11}$$

and the fact that $\left(\sum_{j=1}^{m} |\xi_j|^p \right)^{1/p}$ is decreasing in p [13, Lemma 1.1].

Theorem 2.1 *1.* $S_{n,p}(\omega,\tau) = S_{n,p}(|\omega|,|\tau|)$*;*

2. In $|\omega|$*,* $S_{n,p}(\omega,\tau)$ *is decreasing while* $|\omega|^n S_{n,p}(\omega,\tau)$ *is increasing;*

3. If $|\tau| \geq 1$*,* $S_{n,p}(\omega,\tau)$ *is increasing in* $|\tau|$*.*

Proof: Because $T_n(t)$ is either even or odd, $T_n^{(j)}(t)$ is either even or odd. Thus $|T_n^{(j)}(t)| = |T_n^{(j)}(|t|)|$. With (2.5), it can be seen that

$$\frac{d^j}{dx^j}T_n(x;\omega,\tau) \equiv \frac{d^j}{dx^j}T_n(x/\omega+\tau) = \omega^{-j}\frac{d^j}{dt^j}T_n(t)$$

and thus

$$a_{jn}(\omega,\tau) = \frac{1}{j!}\,\frac{d^j}{dx^j}T_n(x;\omega,\tau)\bigg|_{x=0} = \omega^{-j}\frac{1}{j!}T_n^{(j)}(\tau) \tag{2.12}$$

from which it follows that

$$S_{n,p}(\omega,\tau) = \left(\sum_{j=0}^{n}\left|\omega^{-j}\frac{1}{j!}T_n^{(j)}(\tau)\right|^p\right)^{1/p} = S_{n,p}(|\omega|,|\tau|), \tag{2.13}$$

and that as $|\omega|$ increases, $S_{n,p}(\omega,\tau)$ decreases and $|\omega|^n S_{n,p}(\omega,\tau)$ increases. This proves Items 1 and 2. For Item 3, we claim that $\left|T_n^{(j)}(t)\right|$ increases as t does for $t \geq 1$, this is because the zeros of $T_n^{(j)}(t)$ are all within $(-1,1)$, and thus $\mathsf{sign}(T_n^{(j)}(t)) = \mathsf{sign}(t^{n-j}) = 1$ for $t \geq 1$ and therefore for $t \geq 1$, $\left|T_n^{(j)}(t)\right| = T_n^{(j)}(t)$ is increasing. ∎

Lemma 2.1 *For* $T_n(x;1,0) \equiv T_n(t)$*, i.e.,* $\omega = 1$ *and* $\tau = 0$ *in* (2.5)*, we have*

1. $a_{n-1\,n} = a_{n-3\,n} = \cdots = 0$*, i.e.,* $T_n(t) = a_{nn}t^n + a_{n-2\,n}t^{n-2} + a_{n-4\,n}t^{n-4} + \cdots$*;*
2. $\mathsf{sign}(a_{n-2i\,n}) = (-1)^i$*.*

Proof: Item 1 is a consequence of (2.1). We now prove Item 2 by induction. It holds for $n \leq 1$. Assume it holds for $n-1$ or less. For $i \geq 1$

$$a_{n-2i\,n} = 2a_{n-2i-1,n-1} - a_{n-2i,n-2} = 2a_{(n-1)-2i,n-1} - a_{(n-2)-2(i-1),n-2},$$

where commas in some subscripts are inserted as separators for clarity. Thus

$$\begin{aligned}(-1)^i a_{n-2i\,n} &= 2(-1)^i a_{(n-1)-2i,n-1} - (-1)^i a_{(n-2)-2(i-1),n-2} \\ &= 2(-1)^i a_{(n-1)-2i,n-1} + (-1)^{i-1} a_{(n-2)-2(i-1),n-2} \\ &> 0,\end{aligned}$$

and thus it holds for n. By induction, this completes the proof. ∎

Theorem 2.2 *For* $T_n(x;\omega,0) \equiv T_n(x/\omega)$*, i.e.,* $\tau = 0$ *in* (2.5)*, we have*

1. $a_{n-1\,n} = a_{n-3\,n} = \cdots = 0$, *i.e.*, $T_n(x/\omega) = a_{nn}x^n + a_{n-2\,n}x^{n-2} + a_{n-4\,n}x^{n-4} + \cdots$;

2. $\mathsf{sign}(a_{n-2i\,n}) = (-1)^i \mathsf{sign}(\omega^n)$.

3. $S_{n,1}(\omega, 0) = |T_n(\iota/\omega)|$, *where* $\iota = \sqrt{-1}$ *is the* imaginary unit. *Thus*

$$S_{n,1}(\omega,0) = \frac{1}{2}\left[\left(\frac{1}{|\omega|} + \sqrt{1+\frac{1}{|\omega|^2}}\right)^n + \left(\frac{1}{|\omega|} - \sqrt{1+\frac{1}{|\omega|^2}}\right)^n\right], \quad (2.14)$$

$$\sim \frac{1}{2}\left(\frac{1}{|\omega|} + \sqrt{1+\frac{1}{|\omega|^2}}\right)^n. \quad (2.15)$$

Proof: It can be seen that

$$a_{jn}(\omega, 0) \equiv a_{jn}(1,0)/\,\omega^j. \quad (2.16)$$

So Item 1 is a consequence of Item 1 of Lemma 2.1. Now by Item 2 of Lemma 2.1,

$$\mathsf{sign}(a_{n-2i\,n}(\omega,0)\,) = \mathsf{sign}(a_{n-2i\,n}(1,0)\,)\cdot \mathsf{sign}(\omega^{n-2i}) = (-1)^i \mathsf{sign}(\omega^n),$$

which is Item 2 here. By Items 1 and 2, we have

$$S_{n,1}(\omega, 0) = |a_{nn} - a_{n-2\,n} + a_{n-4\,n} - \cdots|,$$

and thus $S_{n,1}(\omega,0) = |T_n(\iota/\omega)|$ and (2.14) by (2.2). This proves Item 3. ∎

Theorem 2.3 *For* $\omega \neq 0$ *and* $|\tau| \geq 1$, *we have*

1. *If* $\tau \geq 1$, *then* $\mathsf{sign}(a_{jn}) = [\mathsf{sign}(\omega)]^j$, *and thus*

$$S_{n,1}(\omega,\tau) = \begin{cases} T_n(\ \ 1;\omega,\tau), & \textit{if } \omega > 0, \\ (-1)^n T_n(-1;\omega,\tau), & \textit{if } \omega < 0; \end{cases}$$

2. *If* $\tau \leq -1$, *then* $\mathsf{sign}(a_{jn}) = (-1)^{n-j}[\mathsf{sign}(\omega)]^j$, *and thus*

$$S_{n,1}(\omega,\tau) = \begin{cases} (-1)^n T_n(-1;\omega,\tau), & \textit{if } \omega > 0, \\ T_n(\ \ 1;\omega,\tau), & \textit{if } \omega < 0; \end{cases}$$

3. *Thus for* $|\tau| \geq 1$,

$$S_{n,1}(\omega,\tau) = T_n\left(\frac{1}{|\omega|} + |\tau|\right) \sim \frac{1}{2}\left[\left(\frac{1}{|\omega|} + |\tau|\right) + \sqrt{\left(\frac{1}{|\omega|} + |\tau|\right)^2 - 1}\right]^n. \quad (2.17)$$

Proof: These are consequences of (2.12) and the fact that $\mathsf{sign}(T_n^{(j)}(t)) = \mathsf{sign}(t^{n-j}) = (-1)^{n-j}$ for $|t| \geq 1$. ∎

REMARK **2.1** Condition $|\tau| \geq 1$ in Theorem 2.3 can be weakened slightly to $|\tau| \geq \cos\frac{\pi}{2n}$ because the zeros of $T_n(t)$ are $t_j = \cos\theta_j$, $\theta_j = \frac{2j-1}{2n}\pi$ $(1 \leq j \leq n)$. Therefore $\mathsf{sign}(T_n^{(j)}(t)) = \mathsf{sign}(t^{n-j})$ (for $|t| \geq \cos\frac{\pi}{2n}$), the key fact that is critical to the proof.

Theorems 2.2 and 2.3 established $S_{n,1}(\omega, \tau)$ exactly in terms of values of $T_n(x; \omega, \tau)$ at $\pm\iota$ or ± 1. The case for $|\tau| < 1$, except when $\tau = 0$, is still left open. This is caused by the irregularities in the signs of a_{jn} for $|\tau| < 1$ and $\tau \neq 0$.

Theorem 2.4 *Let $\omega > 0$ and $n \geq 2$*

1. *$\omega S_{n,1}(\omega, 0)$ is decreasing in ω if $\omega \leq \max\{\sqrt{n-1}, \sqrt{2}\}$ or n is odd;*
2. *$\omega S_{n,1}(\omega, 1)$ is decreasing in ω if $\omega \leq \max\{n-1, \sqrt{2}\}$;*
3. *$\frac{2\omega}{\sqrt{1+2\omega}} S_n(\omega, 1)$ is decreasing if $\omega \leq \sqrt{n-1}(\sqrt{n-1} + \sqrt{n})$.*

Proof: 1) By (2.14), we have

$$\begin{aligned}
\omega S_{n,1}(\omega, 0) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \left(\frac{1}{\omega}\right)^{n-2j-1} \left(1 + \frac{1}{\omega^2}\right)^j & \text{for odd } n, \\
&= \sum_{j=0}^{\lfloor n/2 \rfloor - 1} \binom{n}{2j} \left(\frac{1}{\omega}\right)^{n-2j-1} \left(1 + \frac{1}{\omega^2}\right)^j + \omega \left(1 + \frac{1}{\omega^2}\right)^{n/2} & \text{for even } n.
\end{aligned}$$

So $\omega S_{n,1}(\omega, 0)$ is always decreasing in ω for odd $n \geq 1$. For even n, however, all terms are decreasing in ω, except possibly the last one. But

$$\frac{d}{d\omega} \omega \left(1 + \frac{1}{\omega^2}\right)^{n/2} = -\left(1 + \frac{1}{\omega^2}\right)^{n/2} \cdot \frac{n - \omega^2 - 1}{\omega^2 + 1} \leq 0$$

if $\omega \leq \sqrt{n-1}$. It can be verified directly that $\omega S_{2,1}(\omega, 0) = \omega + 2/\omega$ is decreasing if $\omega \leq \sqrt{2}$.

2) By (2.3) and (2.17), we have

$$\begin{aligned}
\omega S_{n,1}(\omega, 1) = \quad & \omega \left(\frac{1}{\omega} + 1\right)^n \\
& + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{2j} \left(\frac{1}{\omega} + 1\right)^{n-2j} \left(\frac{2}{\omega} + \frac{1}{\omega^2}\right)^{j-1} \left(2 + \frac{1}{\omega}\right).
\end{aligned}$$

All terms are decreasing in ω, except possibly the first one. But

$$\frac{d}{d\omega} \omega \left(\frac{1}{\omega} + 1\right)^n = \left(\frac{1}{\omega} + 1\right)^n \cdot \left(1 - \frac{n-1}{\omega}\right) \leq 0$$

if $\omega \leq n - 1$. It can be verified directly that $\omega S_{2,1}(\omega, 1) = 2/\omega + 4 + \omega$ is decreasing if $\omega \leq \sqrt{2}$.

3) Again by (2.3) and (2.17), we have

$$\begin{aligned}\frac{2\omega}{\sqrt{1+2\omega}}S_n(\omega,1) &= \frac{2\omega}{\sqrt{1+2\omega}}\left(\frac{1}{\omega}+1\right)^n \\ &+\frac{2}{\sqrt{1+2\omega}}\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n}{2j}\left(\frac{1}{\omega}+1\right)^{n-2j}\left(\frac{2}{\omega}+\frac{1}{\omega^2}\right)^{j-1}\left(2+\frac{1}{\omega}\right).\end{aligned}$$

All terms are decreasing in ω, except possibly the first one. But

$$\frac{d}{d\omega}\frac{2\omega}{\sqrt{1+2\omega}}\left(\frac{1}{\omega}+1\right)^n = 2\left(\frac{1}{\omega}+1\right)^n\frac{\omega^2-2(n-1)\omega-(n-1)}{(1+2\omega)^{3/2}(1+\omega)} \le 0$$

if $\omega \le \sqrt{n-1}(\sqrt{n-1}+\sqrt{n})$. ∎

3 ℓ_p-vector and ℓ_p-operator norms

Let $1 \le p \le \infty$. Given $u = (\mu_1\ \mu_2\ \cdots \mu_n)^T \in \mathbb{R}^n$, its ℓ_p-norm is defined as

$$\|u\|_p = \left(\sum_{j=1}^{n}|\mu_j|^p\right)^{1/p},$$

and $\|u\|_\infty = \lim_{p\to\infty}\|u\|_p = \max_j |\mu_j|$. The associated ℓ_p-operator norm of $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_p = \max_{0\neq u\in\mathbb{R}^n}\frac{\|Au\|_p}{\|u\|_p}. \tag{3.1}$$

It is proved that [17]

$$\|A\|_p = \|A^T\|_{p'}.$$

In particular, $\|A\|_\infty = \|A^T\|_1$. All matrix norms are equivalent, and in particular, we have for $1 \le p,\ q \le \infty$

$$\min\{m^{1/p-1/q}, n^{1/q-1/p}\}\,\|A\|_q \le \|A\|_p \le \max\{m^{1/p-1/q}, n^{1/q-1/p}\}\,\|A\|_q. \tag{3.2}$$

To prove this, we cite the following inequality [13, Lemma 1.1]

$$\|u\|_s \le \|u\|_p \le n^{1/p-1/s}\|u\|_s \quad \text{for } 1\le p\le s\le\infty \text{ and } u\in\mathbb{R}^n. \tag{3.3}$$

This follows from Hölder inequality. Now if $p \le q$ and for $u \in \mathbb{R}^n$,

$$\begin{aligned}\|Au\|_q \le \quad \|Au\|_p &\le m^{1/p-1/q}\|Au\|_q,\\ \|u\|_q \le \quad \|u\|_p &\le n^{1/p-1/q}\|u\|_q.\end{aligned}$$

Therefore

$$n^{-(1/p-1/q)}\frac{\|Au\|_q}{\|u\|_q} \le \frac{\|Au\|_p}{\|u\|_p} \le m^{1/p-1/q}\frac{\|Au\|_q}{\|u\|_q} \quad \text{for } p\le q. \tag{3.4}$$

Similarly, we have

$$m^{-(1/q-1/p)}\frac{\|Au\|_q}{\|u\|_q} \le \frac{\|Au\|_p}{\|u\|_p} \le n^{1/q-1/p}\frac{\|Au\|_q}{\|u\|_q} \quad \text{for } p > q. \tag{3.5}$$

Combining (3.4) and (3.5) yields (3.2). In particular if $m = n$, (3.2) becomes

$$n^{-|1/p-1/q|}\,\|A\|_q \le \|A\|_p \le n^{|1/p-1/q|}\,\|A\|_q. \tag{3.6}$$

Another useful inequality due to Kato [15, Page 29] is

$$\|A\|_p \le \|A\|_1^{1/p}\|A\|_\infty^{1/p'}. \tag{3.7}$$

Define p-condition number $\kappa_p(X)$ of nonsingular $X \in \mathbb{R}^{n\times n}$ by

$$\kappa_p(X) = \|X\|_p\|X^{-1}\|_p. \tag{3.8}$$

By (3.6), we see $\kappa_p(X)$ and $\kappa_q(X)$ differ by a factor n^d for some $|d| \le 2$. In particular

$$n^{-2/p}\kappa_\infty(X) \le \kappa_p(X) \le n^{2/p}\kappa_\infty(X). \tag{3.9}$$

4 Vandermonde matrices with translated Chebyshev nodes

The zeros of $T_n(t)$ are called

$$\textit{Chebyshev Nodes: } t_j = \cos\theta_j,\ \theta_j = \tfrac{2j-1}{2n}\pi\ (1 \le j \le n), \tag{4.1}$$

and the zeros of its translated Chebyshev polynomial $T_n(x;\omega,\tau)$ as in (2.4) are called

$$\textit{Translated Chebyshev Nodes: } x_j = \omega(t_j - \tau)\ (1 \le j \le n). \tag{4.2}$$

Given interval $[a, b]$, set

$$\omega = \frac{b-a}{2} > 0, \quad \tau = -\frac{a+b}{b-a}. \tag{4.3}$$

Then the linear transformation $t = x/\omega + \tau$ defined in (2.5) maps $x \in [a, b]$ one-to-one and onto $t \in [-1, 1]$. The inverse transformation is $x = \omega(t - \tau)$.

This section, inspired by the results of Gautschi [8], computes $\kappa_\infty(V)$ whose nodes are the translated Chebyshev nodes for the case $-a = b$ and the case $0 \le a < b$. But we are still unsure how to deal with the general case $a < 0 < b$ but $-a \ne b$.

First we compute $\|V\|_\infty$ for V with the translated Chebyshev nodes $\alpha_j = x_j$. This is relatively easy.

$$\begin{aligned}
\|V\|_\infty &= \max\left\{n, \sum_{j=1}^n |\alpha_j|^{n-1}\right\} && \text{([9, Theorem 2.1])}\\
&= \max\left\{n, \omega^{n-1}\sum_{j=1}^n |\cos\theta_j - \tau|^{n-1}\right\}\\
&= \max\left\{n, \omega^{n-1}\Lambda_n(\tau)\right\}, && (4.4)
\end{aligned}$$

where

$$\Lambda_n(\tau) \stackrel{\text{def}}{=} \sum_{j=1}^{n} |\cos\theta_j - \tau|^{n-1}. \tag{4.5}$$

It can be seen that $\Lambda_n(-\tau) = \Lambda_n(\tau)$. In Appendix B, the asymptotical behaviors for $\Lambda_n(0)$ and $\Lambda(\pm 1)$ are obtained, but its behavior for any other τ is still unknown. Using on the results there, we obtain the following theorem.

Theorem 4.1 *Let $\alpha_j = x_j$ $(1 \le j \le n)$ as in* (4.2) *with* (4.3)*.*

1. *If $-a = b > 0$ (and thus $\omega = b$), then*

$$\|V\|_\infty \sim \max\left\{n,\ \sqrt{\frac{2n}{\pi}}\,\omega^{n-1}\right\} \sim \max\left\{n,\ \sqrt{\frac{2n}{\pi}}\,\alpha^{n-1}\right\}. \tag{4.6}$$

2. *If $0 = a < b$, then*

$$\|V\|_\infty \sim \max\left\{n,\ \sqrt{\frac{n}{\pi}}\,b^{n-1}\right\} \sim \max\left\{n,\ \sqrt{\frac{n}{\pi}}\,\alpha^{n-1}\right\}. \tag{4.7}$$

Proof: For $-a = b > 0$, $\tau = 0$. By (B.2)

$$\omega^{n-1}\Lambda_n(0) \sim \omega^{n-1}\sqrt{\frac{2n}{\pi}} \sim \sqrt{\frac{2n}{\pi}}\,\alpha^{n-1}\left(\cos\frac{\pi}{2n}\right)^{-(n-1)} \sim \sqrt{\frac{2n}{\pi}}\,\alpha^{n-1},$$

since $-(n-1)\ln\cos\frac{\pi}{2n} \sim \frac{(n-1)\pi^2}{8n^2} \to 0 \Rightarrow \left(\cos\frac{\pi}{2n}\right)^{-(n-1)} \sim 1$. For $0 = a < b$, $\tau = -1$. By (B.4) and since $\Lambda_n(-1) = \Lambda_n(1)$,

$$\begin{aligned}\omega^{n-1}\Lambda_n(1) &\sim \omega^{n-1}\sqrt{\frac{n}{\pi}}\,2^{n-1} = b^{n-1}\sqrt{\frac{n}{\pi}} \\ &= \sqrt{\frac{n}{\pi}}\,\alpha^{n-1}\left(\frac{2}{1+\cos\frac{\pi}{2n}}\right)^{-(n-1)} \sim \sqrt{\frac{n}{\pi}}\,\alpha^{n-1},\end{aligned}$$

since $-(n-1)\ln\frac{2}{1+\cos\frac{\pi}{2n}} \sim -\frac{(n-1)\pi^2}{16n^2} \to 0 \Rightarrow \left(\frac{2}{1+\cos\frac{\pi}{2n}}\right)^{-(n-1)} \sim 1.$ ∎

In both cases $-a = b$ or $0 = a < b$, $\sum_{j=1}^{n}|x_j|^{n-1} = \mathcal{O}(\sqrt{n}\,\alpha^{n-1})$. But will this be also true for arbitrary interval $[a, b]$? We do not know. Nevertheless, we expect at least this be true for all $a < 0 < b$. Figure 4.1 plots the ratio for various $[a, b]$ as n varies, and it supports the following conjecture.

Conjecture 4.1 *For the Chebyshev translated nodes x_j, $\sum_{j=1}^{n}|x_j|^{n-1} = \mathcal{O}(\sqrt{n}\,\alpha^{n-1})$ holds for $a \le 0 \le b$.*

Conjecture 4.1 has already been proved for the cases $-a = b$ or $a = 0$ or $b = 0$ by the proofs of Theorem 4.1.

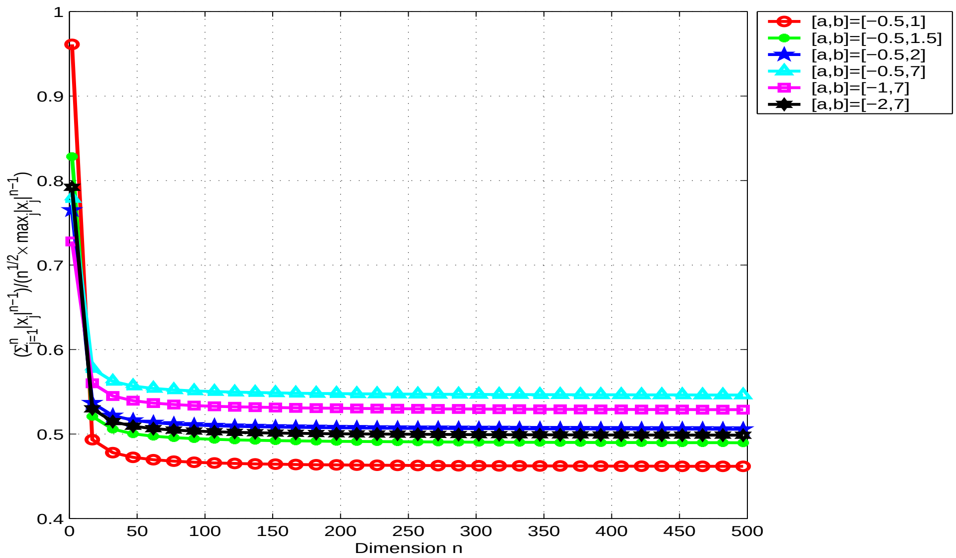


Figure 4.1: Ratios $\frac{\sum_{j=1}^{n}|x_j|^{n-1}}{\sqrt{n}\,\alpha^{n-1}}$ for several $[a,b]$ with $a<0<b$ and $-a\neq b$ as n varies.

Theorem 4.2 (Gautschi, 1975) *If $\alpha_j+\alpha_{n+1-j}=0$ for all $1\le j\le n$, then*

$$\|V^{-1}\|_\infty = \frac{|f(\iota)|}{\min_{\alpha_j\ge 0}\left\{\frac{1+\alpha_j^2}{1+\alpha_j}|f'(\alpha_j)|\right\}}, \tag{4.8}$$

where

$$f(x)=\prod_{j=1}^{n}(x-\alpha_j). \tag{4.9}$$

Theorem 4.3 *Suppose $-a=b>0$, and let $\alpha_j=x_j$ $(1\le j\le n)$ as in* (4.2) *with* (4.3)*. Then V is a $V_{\rm sym}$, and*

$$\omega\min\left\{1,\frac{1+\omega}{1+\omega^2}\right\}\frac{1}{n}\le\frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega,0)}\le\omega\max\left\{1,\frac{1+\omega}{1+\omega^2}\right\}\frac{3^{3/4}}{2n}, \tag{4.10}$$

where the first inequality is valid for $n\ge 3$ only.

Proof: With $\alpha_j=x_j$ for all j, $T_n(x;\omega,\tau)$ and $f(x)$ differ by a constant factor, and thus $f(x)=(1/a_{nn})T_n(x;\omega,\tau)$, where a_{nn} is as defined by (2.6). So (4.8) remains true with $f(x)$ replaced by $T_n(x;\omega,\tau)$. Now notice

$$\begin{aligned}|T_n(-1;\omega,\tau)| &= S_{n,1}(\omega,\tau), && \text{(because } \tau<-1 \text{ and by Theorem 2.3)}\\ |T_n'(x_j;\omega,\tau)| &= \frac{1}{\omega}\,|T_n'(t_j)|=\frac{n}{\sin\theta_j},\end{aligned}$$

to get

$$\frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega,0)} = \frac{\omega}{\min_{t_j\ge 0}\left\{\frac{1+\omega^2 t_j^2}{1+\omega t_j}\,|T_n'(t_j)|\right\}}. \tag{4.11}$$

We now estimate the denominator, but first notice that

$$\begin{aligned}\Delta \overset{\text{def}}{=} \min_{t_j\ge 0}\left\{\frac{1+\omega^2 t_j^2}{1+\omega t_j}\,|T_n'(t_j)|\right\} &= \min_{t_j\ge 0}\left\{\frac{1+\omega^2\cos^2\theta_j}{1+\omega\cos\theta_j}\,\frac{n}{\sin\theta_j}\right\} \\ &\ge n \min_{0\le\theta\le\pi/2}\left\{\frac{1+\omega^2\cos^2\theta}{1+\omega\cos\theta}\,\frac{1}{\sin\theta}\right\}.\end{aligned} \tag{4.12}$$

We can show for $0\le\theta\le\pi/2$

$$\min\left\{1,\frac{1+\omega^2}{1+\omega}\right\}\frac{1+\cos^2\theta}{1+\cos\theta} \le \frac{1+\omega^2\cos^2\theta}{1+\omega\cos\theta} \le \max\left\{1,\frac{1+\omega^2}{1+\omega}\right\}\frac{1+\cos^2\theta}{1+\cos\theta} \tag{4.13}$$

by differentiating

$$\frac{1+\omega^2\cos^2\theta}{1+\omega\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos^2\theta}$$

with respect to $\cos\theta$. Next we can show

$$\min_{0\le\theta\le\pi/2}\frac{1+\cos^2\theta}{1+\cos\theta}\cdot\frac{1}{\sin\theta} = 2\times 3^{-3/4} \tag{4.14}$$

by elementary calculation [8, page 345], and finally

$$\begin{aligned}\min_{0\le\theta_j\le\pi/2}\frac{1+\cos^2\theta_j}{1+\cos\theta_j}\cdot\frac{1}{\sin\theta_j} &\le \left.\frac{1+\cos^2\theta_j}{1+\cos\theta_j}\cdot\frac{1}{\sin\theta_j}\right|_{j=\frac{n+1}{2}} \quad \text{(for odd } n\text{)} \\ &= 1, \\ \min_{0\le\theta_j\le\pi/2}\frac{1+\cos^2\theta_j}{1+\cos\theta_j}\cdot\frac{1}{\sin\theta_j} &\le \left.\frac{1+\cos^2\theta_j}{1+\cos\theta_j}\cdot\frac{1}{\sin\theta_j}\right|_{j=\frac{n}{2}} \quad \text{(for even } n\text{)} \\ &= \frac{1+\sin^2\frac{\pi}{2n}}{1+\sin\frac{\pi}{2n}}\cdot\frac{1}{\cos\frac{\pi}{2n}} \\ &\le 1. \quad \text{(if } n\ge 3\text{)}\end{aligned}$$

Together with (4.14), they imply

$$2\times 3^{-3/4} \le \min_{0\le\theta_j\le\pi/2}\frac{1+\cos^2\theta_j}{1+\cos\theta_j}\cdot\frac{1}{\sin\theta_j} \le 1, \tag{4.15}$$

with the last inequality true for $n\ge 3$ only. Putting (4.12), (4.13), and (4.15) together, we have

$$\min\left\{1,\frac{1+\omega^2}{1+\omega}\right\}\frac{2n}{3^{3/4}} \le \Delta \le \max\left\{1,\frac{1+\omega^2}{1+\omega}\right\}n. \tag{4.16}$$

(4.10) is a direct consequence of (4.11), (4.12), and (4.16) above. ∎

Theorem 4.4 (Gautschi, 1975) *If $\alpha_j \geq 0$ for all $1 \leq j \leq n$, then*

$$\|V^{-1}\|_\infty = \frac{|f(-1)|}{\min_{1\leq j\leq n}\{(1+\alpha_j)|f'(\alpha_j)|\}}, \tag{4.17}$$

where $f(x)$ is defined as in (4.9).

Theorem 4.5 *Suppose $0 \leq a < b$, and let $\alpha_j = x_j$ $(1 \leq j \leq n)$ as in* (4.2) *with* (4.3). *Then*

$$\frac{\frac{b-a}{2}\cos\frac{\pi}{2n}}{n\left(1+\frac{a+b}{2}\right)} \leq \frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega,\tau)} \leq \frac{b-a}{2n\sqrt{(1+b)(1+a)}}. \tag{4.18}$$

Proof: By similar reasoning to the beginning of the proof for Theorem 4.3, we have

$$\frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega,\tau)} = \frac{\omega}{\min_{1\leq j\leq n}\{[1+\omega(t_j-\tau)]|T_n'(t_j)|\}},$$

and

$$\begin{aligned}\min_{1\leq j\leq n}\left\{[1+\omega(t_j-\tau)]|T_n'(t_j)|\right\} &= \min_{1\leq j\leq n}\left\{\left(1+\frac{a+b}{2}+\frac{b-a}{2}\cos\theta_j\right)\frac{n}{\sin\theta_j}\right\}\\ &\geq \min_{|\theta|\leq\pi/2}\left\{\left(1+\frac{a+b}{2}+\frac{b-a}{2}\cos\theta\right)\frac{n}{\sin\theta}\right\},\end{aligned}$$

where the last $\min_{|\theta|\leq\pi/2}\{\cdots\}$ achieves its minimum $n\sqrt{(1+b)(1+a)}$ at

$$\cos\theta_{\min} = -\frac{(b-a)/2}{1+(a+b)/2} < 0.$$

This yields the second inequality in (4.18). The first inequality holds because

$$\begin{aligned}\min_{1\leq j\leq n}\left\{[1+\omega(t_j-\tau)]|T_n'(t_j)|\right\} &\leq \left(1+\frac{a+b}{2}+\frac{b-a}{2}\cos\theta_j\right)\frac{n}{\sin\theta_j}\bigg|_{j=\frac{n+1}{2}} \quad \text{(for odd } n\text{)}\\ &= \left(1+\frac{a+b}{2}\right)n,\\ \min_{1\leq j\leq n}\left\{[1+\omega(t_j-\tau)]|T_n'(t_j)|\right\} &\leq \left(1+\frac{a+b}{2}+\frac{b-a}{2}\cos\theta_j\right)\frac{n}{\sin\theta_j}\bigg|_{j=\frac{n}{2}} \quad \text{(for even } n\text{)}\\ &\leq \left(1+\frac{a+b}{2}\right)\frac{n}{\cos\frac{\pi}{2n}}.\end{aligned}$$

This completes the proof. ∎

Theorems 4.3 and 4.5 say that for V with translated Chebyshev nodes on $[a,b]$ if $-a=b$ or $0\leq a<b$ or $a<b\leq 0$, then

$$\frac{n\|V^{-1}\|_\infty}{S_{n,1}(\omega,\tau)} = \mathcal{O}(1). \tag{4.19}$$

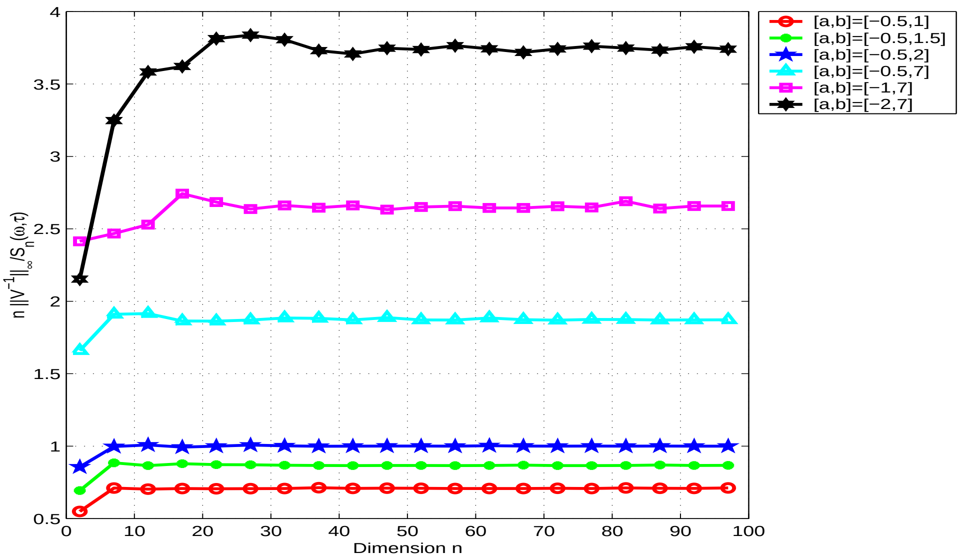


Figure 4.2: Ratio $\frac{n\|V^{-1}\|_\infty}{S_{n,1}(\omega,\tau)}$ for several $[a,b]$ with $a<0<b$ and $-a\neq b$ as n varies.

(The case $[a,b]$ for $a<b\leq 0$ can be turned into $[-b,-a]$, a case that is covered by Theorem 4.5.) But what happens when $a<0<b$ and $-a\neq b$? Figure 4.2 plots the ratio for various $[a,b]$ as n varies, where ratios were carefully performed by MAPLE. This figure supports the following conjecture.

Conjecture 4.2 (4.19) *would still be true for* $a<0<b$ *but* $-a\neq b$.

Theorem 4.6 *Let* $\alpha_j=x_j$ $(1\leq j\leq n)$ *as in* (4.2) *with* (4.3).

1. *If* $-a=b>0$, *then*

$$\min_{-a=b}\kappa_\infty(V)\leq\frac{3^{3/4}}{2}b_{\text{opt}}\,S_{n,1}(b_{\text{opt}},0)\sim\frac{3^{3/4}}{2}\left(\frac{2}{\pi}\right)^{\sqrt{2}/4}\frac{(1+\sqrt{2}\,)^n}{2n^{\sqrt{2}/4}},\tag{4.20}$$

where $b_{\text{opt}}\equiv\omega_{\text{opt}}=(n/\Lambda_n(0))^{1/(n-1)}$.

2. *If* $0=a<b$, *then*

$$\min_{0=a<b}\kappa_\infty(V)\leq\frac{b^+_{\text{opt}}}{2\sqrt{1+b^+_{\text{opt}}}}S_{n,1}(b^+_{\text{opt}}/2,1)\sim\frac{\sqrt{2}\,(1+\sqrt{2})^{2n}}{4(n\pi)^{\sqrt{2}/4}},\tag{4.21}$$

where $b^+_{\text{opt}}/2\equiv\omega^+_{\text{opt}}=(n/\Lambda_n(1))^{1/(n-1)}$.

Proof: 1) Assume $-a=b$. By Theorem 4.3, we have

$$\kappa_\infty(V)\leq\max\left\{1,\frac{1+\omega}{1+\omega^2}\right\}\frac{3^{3/4}}{2n}\times\max\{n\omega S_{n,1}(\omega,0),\omega^n\Lambda_n(0)S_{n,1}(\omega,0)\}\tag{4.22}$$

whose right-hand side we would like to minimize, ideally. But that's not easy; so instead we shall attempt to minimize the factor

$$\Delta = \max\{n\omega S_{n,1}(\omega, 0), \omega^n \Lambda_n(0) S_{n,1}(\omega, 0)\}.$$

This is is a good compromise because the growth of $S_{n,1}(\omega, 1)$ dominates that of the factor before "$\times$". To do so, we notice:

1. By Theorem 2.1 $\omega^n S_{n,1}(\omega, 0)$ is increasing in ω.
2. By Theorem 2.4 $\omega S_{n,1}(\omega, 0)$ is decreasing in ω if $\omega \le \max\{\sqrt{n-1}, \sqrt{2}\}$ or n is odd.

So for $\omega \le \max\{\sqrt{n-1}, \sqrt{2}\}$, Δ is minimized when the two quantities within $\max\{\cdots\}$ are equal, i.e, at

$$\omega^{n-1} = n/\Lambda_n(0) \quad \Rightarrow \quad \omega = (n/\Lambda_n(0))^{1/(n-1)} \equiv \omega_{\text{opt}},$$

at which $\Delta = n\omega_{\text{opt}} S_{n,1}(\omega_{\text{opt}}, 0)$. Notice that $\sqrt{2} \ge (n/\Lambda_n(0))^{1/(n-1)} \to 1$ (as $n \to \infty$), and on the other hand, for $\omega \ge \max\{\sqrt{n-1}, \sqrt{2}\}$, $\Delta \ge \Lambda_n(0) 2^{n/2} S_{n,1}(\sqrt{2}, 0)$ which is much bigger than $n\omega_{\text{opt}} S_{n,1}(\omega_{\text{opt}}, 0)$ for large n. Therefore[1] Δ is minimized at $\omega = \omega_{\text{opt}}$ for large n. Substitute $\omega = \omega_{\text{opt}}$ into the right-hand side of (4.22) to get (4.20), except the asymptotic part for which we refer to (6.13).

2) By Theorem 4.5, we have

$$\kappa_\infty(V) \le \frac{b}{2n\sqrt{1+b}} \cdot \frac{1}{\omega} \times \max\{n\omega S_{n,1}(\omega, 1), \omega^n \Lambda_n(1) S_{n,1}(\omega, 1)\} \tag{4.23}$$

whose right-hand side we would like to minimize, ideally. But again instead, we shall attempt to minimize the factor

$$\Delta = \max\{n\omega S_{n,1}(\omega, 1), \omega^n \Lambda_n(1) S_{n,1}(\omega, 1)\}.$$

This is is a good compromise because the growth of $S_{n,1}(\omega, 1)$ dominates that of the factor before "$\times$". To do so, we notice:

1. By Theorem 2.1 $\omega^n S_{n,1}(\omega, 1)$ is increasing in ω.
2. By Theorem 2.4 $\omega S_{n,1}(\omega, 1)$ is decreasing in ω if $\omega \le \max\{n-1, \sqrt{2}\}$.

So for $\omega \le \max\{n-1, \sqrt{2}\}$, Δ is minimized when the two quantities within $\max\{\cdots\}$ are equal, i.e, at

$$\omega^{n-1} = n/\Lambda_n(1) \quad \Rightarrow \quad \omega = (n/\Lambda_n(1))^{1/(n-1)} \equiv \omega_{\text{opt}}^{+}$$

at which $\Delta = n\omega_{\text{opt}}^{+} S_{n,1}(\omega_{\text{opt}}^{+}, 1)$. Notice that $1 \ge (n/\Lambda_n(1))^{1/(n-1)} \to 1/2$ (as $n \to \infty$), and on the other hand, for $\omega \ge \max\{n-1, \sqrt{2}\}$, $\Delta \ge \Lambda_n(1) 2^{n/2} S_{n,1}(\sqrt{2}, 1)$ which is much

[1]It appears that Δ is minimized at $\omega = \omega_{\text{opt}}$ for all $n \ge 2$. We know this is true for large n through an asymptotic analysis, and for small n through direct verifications (which can only be done for finitely many n, unfortunately), but a rigorous mathematical proof eludes me.

n	2	3	4	5	10	20	40
$b_{\rm opt}$	$\sqrt{2}$	$\sqrt{2}$	1.33	1.28	1.16	1.09	1.05
$b^+_{\rm opt}$	2	$\sqrt{8/3}$	1.47	1.38	1.21	1.11	1.06

Table 4.1: $b_{\rm opt}$ and $b^+_{\rm opt}$ in Theorem 4.6

bigger than $n\omega^+_{\rm opt}S_{n,1}(\omega^+_{\rm opt},1)$ for large n. Therefore[2] Δ is minimized at $\omega=\omega^+_{\rm opt}$ for large n. Substitute $\omega=\omega^+_{\rm opt}$ into the right-hand side of (4.23) to get (4.21), except the asymptotic part for which we refer to (7.17). ∎

Table[3] 4.1 lists the first few $b_{\rm opt}$ and $b^+_{\rm opt}$ as given in Theorem 4.6.

5 Condition numbers of Vandermonde matrices – a general theorem

We shall start by establishing a general theorem on $\kappa_p(V)$ for

$$a\le\min_j\alpha_j\le\max_j\alpha_j\le b. \tag{5.1}$$

The case $a=b$ is of no interest, because then V is of rank 1 and thus $\kappa_p(V)=+\infty$ (unless $n=1$). There are lots of ways to realize (5.1), and it is tempting to always let $a=\min_j\alpha_j$ and $b=\max_j\alpha_j$; but that may not be always possible for theorems that require $-a=b$. Recall ω and τ defined by (4.3), and define

$$\alpha\stackrel{\rm def}{=}\max_j|\alpha_j|. \tag{5.2}$$

Lemma 5.1

$$\max\{n,n\alpha^{n-1}\}\ge\|V\|_1=\sum_{j=1}^n|\alpha|^{j-1}\ge\max\{1,\alpha^{n-1}\}, \tag{5.3}$$

$$\max\{n,n\alpha^{n-1}\}\ge\|V\|_\infty=\max\left\{n,\sum_{j=1}^n|\alpha_j|^{n-1}\right\}\ge\max\{n,\alpha^{n-1}\}, \tag{5.4}$$

$$\max\{n,n\alpha^{n-1}\}\ge\|V\|_p\ge\max\{n^{1/p'},\alpha^{n-1}\}, \tag{5.5}$$

$$\|V^{-1}\|_p\ge\frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}\ge\frac{S_{n-1,1}(\omega,\tau)}{n}. \tag{5.6}$$

[2] It appears that Δ is minimized at $\omega=\omega^+_{\rm opt}$ for all $n\ge2$, but again a rigorous mathematical proof is elusive.

[3] Through numerical computations, it seems that both $b_{\rm opt}$ and $b^+_{\rm opt}$ monotonically decrease to 1. It would be of interest to prove it rigorously.

Proof: The middle equation in (5.3) is due to the known formula for $\|\cdot\|_1$ [5, Page 22] and the middle equation in (5.4) is due to [9, Theorem 2.1]. Inequalities in (5.3) and (5.4) are their consequences. Let e_j be the jth column of the $n \times n$ identity matrix. Then

$$\|V\|_p = \|V^T\|_{p'} \geq \left\{ \begin{array}{l} \|V^T e_1\|_{p'} = n^{1/p'}, \\ \|V^T e_n\|_{p'} \geq \alpha^{n-1}. \end{array} \right.$$

This yields the second inequality in (5.5). Use (3.7), (5.3), and (5.4) to arrive at the first inequality there.

We now show (5.6). Let v be the vector of the coefficients of $T_{n-1}(x;\omega,\tau) \equiv T_{n-1}(x/\omega+\tau)$, i.e., $v = (a_{0,n-1}\ a_{1,n-1}\ \cdots\ a_{n-1,n-1})^T$. Then

$$V^T v = (T_{n-1}(\alpha_1/\omega+\tau)\ T_{n-1}(\alpha_2/\omega+\tau)\ \cdots\ T_{n-1}(\alpha_n/\omega+\tau))^T$$

which yields $\|V^T v\|_{p'} \leq n^{1/p'}$ because $|T_{n-1}(x/\omega+\tau)| \leq 1$ for $x \in [a,b]$. We therefore have

$$\|V^{-1}\|_p = \|V^{-T}\|_{p'} \geq \frac{\|v\|_{p'}}{\|V^T v\|_{p'}} \geq \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}} \geq \frac{S_{n-1,1}(\omega,\tau)}{n},$$

by (2.9). ∎

Remark **5.1** The last inequality in (5.6) can be improved when $-a = b$ (and thus $\tau = 0$) because of (2.10): *if $a = -b$, then*

$$\|V^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega,0)}{n^{1/p'}} \geq \left(\frac{n}{\lceil n/2 \rceil}\right)^{1/p} \frac{S_{n-1,1}(\omega,0)}{n}. \tag{5.7}$$

An interesting observation also for the case $-a = b$ (and thus $\tau = 0$) is as follows. For even n, $T_{n-1}(x;\omega,0)$ is odd, i.e., $a_{0,n-1} = a_{2,n-1} = \cdots = 0$, and thus only the odd rows of V got picked up by $V^T v$, completely discarding all the even rows; while for odd n, for the same reason only the even rows of V got picked up by $V^T v$, completely discarding all the odd rows. Therefore, we conclude that *if $-a = b$, then*

$$\|\widehat{V}^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega,0)}{n^{1/p'}} \geq \frac{S_{n-1,1}(\omega,0)}{n} \tag{5.8}$$

for any $\widehat{V}$ that has the same odd rows as V if n is even or the same even rows as V if n is odd. This remark applies to some of the later lemmas and theorems, too.

Theorem 5.1

$$\kappa_p(V) \geq \max\left\{S_{n-1,p'}(\omega,\tau), \frac{\alpha^{n-1} S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}\right\}, \tag{5.9}$$

$$\geq \max\left\{\frac{S_{n-1,1}(\omega,\tau)}{n^{1/p}}, \frac{\alpha^{n-1} S_{n-1,1}(\omega,\tau)}{n}\right\}. \tag{5.10}$$

Proof: It is an immediate consequence of Lemma 5.1 and (2.9). ∎

This is the most general theorem of this paper for a lower bound on $\kappa_p(V)$. It is its various applications combined with results in Section 4 that lead to many interesting asymptotically optimal bounds. There are at least two different ways to apply Theorem 5.1 to any given V:

1. Take $a = \min_j \alpha_j$ and $b = \alpha$ and then compute the right-hand side of (5.9) or (5.10). But unless $a \geq 0$ or $-a = b$ by coincidence, we may have to compute $S_{n-1,1}(\omega, \tau)$ in a brute force way because no explicit formula has been found yet. In this case, both a and b are nodes of V.

2. Take $-a = b = \alpha$ (and thus $\omega = \alpha$ and $\tau = 0$) and then use the explicit formula for $S_{n-1,1}(\alpha, 0)$ to compute the right-hand side of (5.10). In this case, one of a and b is guaranteed to be a node for V.

Remark **5.2** Another consequence of Lemma 5.1 is

$$\kappa_p(V) \geq \begin{cases} S_{n-1,p'}(\omega, \tau), & \text{if } \alpha \leq 1, \\ \dfrac{\alpha^{n-1} S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}, & \text{if } \alpha > 1 \end{cases} \tag{5.11}$$

upon noticing that $\|V\|_p \geq n^{1/p'}$ if $\alpha \leq 1$, and $\|V\|_p \geq \alpha^{n-1}$ if $\alpha > 1$. (5.11) is slightly weaker than (5.9). But we still state it here for later use in the proof of Theorem 6.3.

6 Condition numbers of Vandermonde matrices whose nodes are in $[a, b]$ with $-a = b$

In this section, we apply Theorem 5.1 and (5.11) to the case $-a = b$.

Theorem 6.1

$$\kappa_p(V) \geq \begin{cases} S_{n-1,p'}(\alpha, 0), & \textit{if } \alpha \leq n^{1/[p'(n-1)]}, \\ \dfrac{\alpha^{n-1} S_{n-1,p'}(\alpha, 0)}{n^{1/p'}}, & \textit{if } \alpha > n^{1/[p'(n-1)]}. \end{cases} \tag{6.1}$$

Proof: Apply Theorem 5.1 to the case $-a = b = \alpha$ (and thus $\omega = \alpha$ and $\tau = 0$) to get

$$\kappa_p(V) \geq \max\left\{ S_{n-1,p'}(\alpha, 0), \frac{\alpha^{n-1} S_{n-1,p'}(\alpha, 0)}{n^{1/p'}} \right\}. \tag{6.2}$$

We shall minimize the right-hand side of (6.2). By Theorem 2.1, the first quantity within $\max\{\cdots\}$ in (6.2) is decreasing in α; while the second one is increasing in α. Therefore the right-hand side of (6.2) achieves its minimum when the two equal, i.e.,

$$n^{1/p'} = \alpha^{n-1} \quad \Rightarrow \quad \alpha = n^{1/[p'(n-1)]}$$

which yields (6.1). ∎

Theorem 6.2

$$S_{n-1,p'}(n^{1/[p'(n-1)]},0) \le \min_{\alpha_j} \kappa_p(V) \le \min_{\alpha_j} \kappa_p(V_{\rm sym}) \le n^{1/p}\left(3^{3/4}/2\right)\cdot S_{n,1}(1,0). \tag{6.3}$$

Proof: The right-hand side of (6.2), as a function of α, achieves its minimum at $\alpha = n^{1/[p'(n-1)]}$. That gives the first inequality. The second inequality is true because $\{V_{\rm sym}\}$ is a subset of all Vandermonde matrices. We now prove the third one. To this end, consider $V = V_{\rm sym}$ with Chebyshev nodes t_j. Then $\|V\|_1 \le n$ and $\|V\|_\infty = n$, and thus by (3.7)

$$\|V\|_p \le \|V\|_1^{1/p}\|V\|_\infty^{1/p'} \le n.$$

Apply Theorem 4.3 to the case $-a = b = 1 = \omega$ to get $\|V^{-1}\|_\infty \le \left(3^{3/4}/2\right)\cdot n^{-1}S_{n,1}(1,0)$ and then to get

$$\|V^{-1}\|_p \le n^{1/p}\|V^{-1}\|_\infty \le n^{1/p}\left(3^{3/4}/2\right)\cdot n^{-1}S_{n,1}(1,0).$$

So for this $V_{\rm sym}$, $\kappa_p(V_{\rm sym}) \le n^{1/p}\left(3^{3/4}/2\right)\cdot S_{n,1}(1,0)$, as needed. ∎

We include $\min_{\alpha_j} \kappa_p(V_{\rm sym})$ in (6.3) mainly because Vandermonde matrices with symmetric nodes were heavily studied by Gautschi and his coauthor [8, 9, 12]. Moreover, assuming that the optimally conditioned V is unique, Gautschi [9] showed that the optimally conditioned V must have symmetric nodes. However, whether this uniqueness assumption is true or not is an open problem.

The third inequality in (6.3) was proved by simply picking a special V with Chebyshev nodes. This turns out to be good enough, as we shall see later, in yielding the correct asymptotic speed in our notation $\mathcal{O}_n$, but it does not produce the best possible factor n^d hidden in the notation. For $p = \infty$, however, a tighter upper bound is possible by using the V with the translated Chebyshev nodes in $[-b_{\rm opt}, b_{\rm opt}]$, where $b_{\rm opt} = (n/\Lambda_n(0))^{1/(n-1)}$ as in Theorem 4.6. Of course, one may use this V for all p, but doing so will not only lead to a more complicated bound but also the resulted bound may not be much better due to more complicated estimation of $\|V\|_p$. For this reason, we shall state a sharper version of Theorem 6.2 for $p = \infty$ only as follows. It is sharper because of $b_{\rm opt} > 1$ and Theorem 2.4.

Theorem 6.2′

$$S_{n-1,1}(n^{1/(n-1)},0) \le \min_{\alpha_j} \kappa_\infty(V) \le \min_{\alpha_j} \kappa_\infty(V_{\rm sym}) \le \left(3^{3/4}/2\right) b_{\rm opt}\, S_{n,1}(b_{\rm opt},0), \tag{6.4}$$

where $b_{\rm opt} = (n/\Lambda_n(0))^{1/(n-1)}$.

Proof: It is a consequence of Theorem 6.2 for $p = \infty$ and Theorem 4.6. ∎

In what follows, we shall establish theorems that are of the same spirit as Theorems 6.2 and 6.2′ but with α subject to a constraint.

Theorem 6.3 *Let* $\delta > 0$ *and set* $\delta' = \delta/\cos\frac{\pi}{2n}$. *If* $\delta \le 1$, *then*

$$S_{n-1,p'}(\delta,0) \le \min_{\alpha\le\delta} \kappa_p(V) \le \min_{\alpha=\delta} \kappa_p(V) \le n^{1/p}\frac{(\sqrt{2}+1)3^{3/4}}{4}\,\delta'\, S_{n,1}(\delta',0); \tag{6.5}$$

and if $\delta > 1$*, then*

$$\frac{\delta^{n-1} S_{n-1,p'}(\delta,0)}{n^{p'}} \le \min_{\alpha=\delta} \kappa_p(V) \le n^{1/p} \frac{3^{3/4}\left[\cos\frac{\pi}{2n}\right]^{n-1}}{2} (\delta')^n S_{n,1}(\delta',0), \tag{6.6}$$

$$S_{n-1,p'}(n^{1/[p'(n-1)]},0) \le \min_{\alpha\le\delta} \kappa_p(V) \le n^{1/p}\left(3^{3/4}/2\right) \cdot S_{n,1}(1,0). \tag{6.7}$$

Inequalities (6.5)*,* (6.6)*, and* (6.7) *remain valid with* V *replaced by* $V_{\rm sym}$*.*

Proof: 1) Observe that $\{V : \alpha = \delta\} \subset \{V : \alpha \le \delta\}$. So

$$\min_{\alpha\le\delta} \kappa_p(V) \le \min_{\alpha=\delta} \kappa_p(V). \tag{6.8}$$

This is the middle inequality in (6.5).

2) Apply (5.11) to the case $-a = b = \alpha \le \delta \le 1$ (and thus $\omega = \alpha$ and $\tau = 0$) to obtain

$$\kappa_p(V) \ge S_{n-1,p'}(\alpha,0) \ge S_{n-1,p'}(\delta,0)$$

By Theorem 2.1. This gives the first inequality in (6.5).

3) Apply (5.11) to the case $-a = b = \delta = \alpha$ (and thus $\omega = \delta$ and $\tau = 0$) to obtain the first inequality in (6.6).

4) Take $-a = b = \delta/\cos\frac{\pi}{2n} = \delta'$, and $\alpha_j = x_j$ $(1 \le j \le n)$, the translated Chebyshev nodes as in (4.2). Then

$$\tau = 0, \quad \alpha = \max|\alpha_j| = b\cos\frac{\pi}{2n} = \delta,$$

$$\delta \le \omega = b = \delta'.$$

Theorem 4.3 says that for the V with those nodes

$$\frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega,0)} \le \omega \max\left\{1, \frac{1+\omega}{1+\omega^2}\right\} \frac{3^{3/4}}{2n} \le \begin{cases} \delta' \dfrac{(\sqrt{2}+1)3^{3/4}}{4n}, & \text{if } \delta \le 1, \\ \delta' \dfrac{3^{3/4}}{2n}, & \text{if } \delta \ge 1, \end{cases}$$

where we have used

$$\max_{\omega>0}\left\{1, \frac{1+\omega}{1+\omega^2}\right\} = \left.\frac{1+\omega}{1+\omega^2}\right|_{\omega=\sqrt{2}-1} = \frac{\sqrt{2}+1}{2}, \quad \max_{\omega\ge1}\left\{1, \frac{1+\omega}{1+\omega^2}\right\} = 1. \tag{6.9}$$

Now employ $\|V\|_p \le n$ if $\delta \le 1$ and $\|V\|_p \le n\delta^{n-1}$ if $\delta \ge 1$, and $\|V^{-1}\|_p \le n^{1/p}\|V^{-1}\|_\infty$ to get the last inequalities in (6.5) and in (6.6).

5) A proof of (6.7) can be done in the same way as for Theorem 6.2.

6) Finally, when V is replaced by $V_{\rm sym}$, the first inequalities in (6.5), (6.6), and (6.7) still hold because any $V_{\rm sym}$ is a Vandermonde matrix. The middle inequality in (6.5) remains

valid, too, because (6.8) is true with V replaced by $V_{\rm sym}$. The last inequalities in (6.5), (6.6), and (6.7) hold because they all were proved by bounding some $\kappa_p(V_{\rm sym})$. ∎

There are stronger versions of (6.6) and (6.7) for $p = \infty$, too, just as we did for Theorem 6.2.

Theorem 6.3′ *Let $\delta > 1$ and set $\delta' = \delta / \cos\frac{\pi}{2n}$. Then*

$$\frac{\delta^{n-1} S_{n-1,1}(\delta, 0)}{n} \leq \min_{\alpha=\delta} \kappa_\infty(V) \leq \frac{3^{3/4}}{2} \frac{\max\{n, \Lambda_n(0)(\delta')^{n-1}\}}{n} \delta' \, S_{n,1}(\delta', 0), \tag{6.10}$$

$$S_{n-1,1}(n^{1/(n-1)}, 0) \leq \min_{\alpha\leq\delta} \kappa_\infty(V) \leq \frac{3^{3/4}}{2} \omega_1 \, S_{n,1}(\omega_1, 0), \tag{6.11}$$

where $\omega_1 = \min\{\delta', (n/\Lambda_n(0))^{1/(n-1)}\}$. It can be seen that $\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}$ for n sufficiently large. Inequalities (6.10)*, and* (6.11) *remain valid with V replaced by $V_{\rm sym}$.*

Proof: Only the second inequalities in (6.10) and (6.11) need proofs. For (6.10), it follows from the proof of Theorem 6.3, upon using $\|V\|_\infty = \max\{n, \omega^{n-1}\Lambda_n(0)\}$ which for large n is proportional to $\sqrt{n}\,\delta^{n-1}$ instead of $\|V\|_\infty \leq n\delta^{n-1}$. The second inequality in (6.11) is obtained by approximately minimizing

$$\frac{3^{3/4}}{2} \frac{\max\{n, \Lambda_n(0)(\alpha')^{n-1}\}}{n} \alpha' \, S_{n,1}(\alpha', 0),$$

subject to $\alpha \leq \delta$, where $\alpha' = \alpha / \cos\frac{\pi}{2n}$. That $\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}$ for n sufficiently large is due to $(n/\Lambda_n(0))^{1/(n-1)} \sim (2\pi/n)^{1/[2(n-1)]} \sim 1$. ∎

We shall now investigate the tightness of the upper bounds and the lower bounds in (6.3) of Theorem 6.2, (6.4) of Theorem 6.2′, (6.5) and (6.6) of Theorem 6.3, and (6.10) and (6.11) of Theorem 6.3′, as well as the asymptotical speeds of $\kappa_p(V)$ minimized over certain set of Vandermonde matrices. We shall do so only for $p = \infty$; for any other p, $S_{n-1,p'}$ in the lower bounds will have to be weakened by using (2.10) so as to apply the same lines of arguments here. Also as far as asymptotical speed is concerned, it suffices to only look into the case $p = \infty$.

For $p = \infty$, $p' = 1$. Since

$$n^{1/(n-1)} = 1 + \frac{\ln(n-1)}{n-1} + \frac{2 + \ln^2(n-1)}{2(n-1)^2} + \cdots,$$

Theorem 6.2 says that $\kappa_\infty(V)$ is no smaller than

$$\begin{aligned} S_{n-1,1}(n^{1/(n-1)}, 0) \;&=\; \frac{(n-1)^{-1/\sqrt{2}}}{2} (1+\sqrt{2})^{n-1} \left[1 + \mathcal{O}\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\ &\quad + (-1)^{n-1} \frac{(n-1)^{1/\sqrt{2}}}{2} (1+\sqrt{2})^{1-n} \left[1 - \mathcal{O}\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\ &\sim\; \frac{(1+\sqrt{2})^{n-1}}{2\,(n-1)^{1/\sqrt{2}}}. \end{aligned} \tag{6.12}$$

On the other hand, $S_{n,1}(1,0) = \frac{1}{2}(1+\sqrt{2}\,)^n + \frac{(-1)^n}{2}(1+\sqrt{2})^{-n}$, and for $b_{\rm opt} = (n/\Lambda_n(0))^{1/(n-1)}$,

$$\begin{aligned} S_{n,1}(b_{\rm opt},0) &= \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \frac{n^{-\sqrt{2}/4}}{2}(1+\sqrt{2})^n \left[1+\mathcal{O}\left(\frac{\ln n)}{n}\right)\right] \\ &\quad +(-1)^n \left(\frac{2}{\pi}\right)^{-\sqrt{2}/4} \frac{n^{\sqrt{2}/4}}{2}(1+\sqrt{2})^n \left[1-\mathcal{O}\left(\frac{\ln n)}{n}\right)\right] \\ &\sim \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \frac{1}{2\,n^{\sqrt{2}/4}}(1+\sqrt{2})^n. \end{aligned} \tag{6.13}$$

Therefore

$$\frac{S_{n,1}(1,0)}{S_{n-1,1}(n^{1/(n-1)},0)} \sim (1+\sqrt{2})\,n^{1/\sqrt{2}}, \tag{6.14}$$

$$\frac{S_{n,1}(b_{\rm opt},0)}{S_{n-1,1}(n^{1/(n-1)},0)} \sim \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} (1+\sqrt{2})\,n^{\sqrt{2}/4}, \tag{6.15}$$

$$\min_{\alpha_j} \kappa_\infty(V_{\rm sym}),\ \min_{\alpha_j} \kappa_\infty(V) = \mathcal{O}_n\left((1+\sqrt{2}\,)^n\right). \tag{6.16}$$

Even though $b_{\rm opt} \sim 1$, $S_{n,1}(b_{\rm opt},0)$ is about $(2/\pi)^{\sqrt{2}/4}\,n^{-\sqrt{2}/4}$ times smaller than $S_{n,1}(1,0)$. We now turn to the bounds in Theorem 6.3. We claim

$$\left(\frac{1+\sqrt{\delta'^2+1}}{1+\sqrt{\delta^{\,2}+1}}\right)^n \sim 1. \tag{6.17}$$

This is because

$$1 \le \frac{1+\sqrt{\delta'^2+1}}{1+\sqrt{\delta^{\,2}+1}} \le \left(\cos\frac{\pi}{2n}\right)^{-1} \quad \Rightarrow \quad 1 \le \left(\frac{1+\sqrt{\delta'^2+1}}{1+\sqrt{\delta^{\,2}+1}}\right)^n \le \left(\cos\frac{\pi}{2n}\right)^{-n} \sim 1$$

since $-n \ln\cos\frac{\pi}{2n} \sim \frac{\pi^2}{8n} \to 0 \Rightarrow \left(\cos\frac{\pi}{2n}\right)^{-n} \sim 1$. With (6.17), we have, by (2.15),

$$\begin{aligned} \frac{\delta'\,S_{n,1}(\delta',0)}{S_{n-1,1}(\delta,0)} &= \frac{\delta'\,S_{n,1}(\delta',0)}{\delta\,S_{n,1}(\delta,0)} \times \frac{\delta\,S_{n,1}(\delta,0)}{S_{n-1,1}(\delta,0)} \\ &\sim \left(\cos\frac{\pi}{2n}\right)^{n-1} \left(\frac{1+\sqrt{\delta'^2+1}}{1+\sqrt{\delta^{\,2}+1}}\right)^n \times (1+\sqrt{\delta^2+1}) \\ &\sim 1+\sqrt{\delta^2+1}, \end{aligned} \tag{6.18}$$

$$\begin{aligned} \frac{\delta'^n\,S_{n,1}(\delta',0)}{\delta^{n-1}\,S_{n-1,1}(\delta,0)} &= \frac{\delta'^n\,S_{n,1}(\delta',0)}{\delta^{\,n}\,S_{n,1}(\delta\,,0)} \times \frac{\delta^{\,n}\,S_{n,1}(\delta\,,0)}{\delta^{n-1}\,S_{n-1,1}(\delta,0)} \\ &\sim \left(\frac{1+\sqrt{\delta'^2+1}}{1+\sqrt{\delta^{\,2}+1}}\right)^n \times (1+\sqrt{\delta^2+1}) \\ &\sim 1+\sqrt{\delta^2+1}. \end{aligned} \tag{6.19}$$

Inequality	Ratio $\sim$		for bounds on
(6.3)	$\frac{(1+\sqrt{2})3^{3/4}}{2}$	$\times \quad n^{1/\sqrt{2}}$	$\min\limits_{\alpha_j}$
(6.4)	$\frac{(1+\sqrt{2})3^{3/4}}{2}\left(\frac{2}{\pi}\right)^{\sqrt{2}/4}$	$\times \quad n^{\sqrt{2}/4}$	
(6.5)	$\frac{(1+\sqrt{2})3^{3/4}}{2}(1+\sqrt{\delta^2+1}\,)$	$\times \quad n^0$	$\min\limits_{\alpha\le\delta}$, $\min\limits_{\alpha=\delta}$ for $\delta \le 1$
(6.6)	$\frac{3^{3/4}}{2}(1+\sqrt{\delta^2+1}\,)$	$\times \quad n^1$	$\min\limits_{\alpha=\delta}$ for $\delta > 1$
(6.10)	$\frac{3^{3/4}}{2}\sqrt{\frac{\pi}{2}}(1+\sqrt{\delta^2+1}\,)$	$\times \quad n^{1/2}$	
(6.7)	$\frac{(1+\sqrt{2})3^{3/4}}{2}$	$\times \quad n^{1/\sqrt{2}}$	$\min\limits_{\alpha\le\delta}$ for $\delta > 1$
(6.11)	$\frac{(1+\sqrt{2})3^{3/4}}{2}\left(\frac{2}{\pi}\right)^{\sqrt{2}/4}$	$\times \quad n^{\sqrt{2}/4}$	

Table 6.1: Ratios of the upper bounds over the lower bounds for $p = \infty$

For the bounds in Theorem 6.3′, we notice

$$\max\{n, \Lambda_n(0)(\delta')^{n-1}\} = \Lambda_n(0)(\delta')^{n-1} \sim \sqrt{\frac{\pi}{2}}\,\sqrt{n}\,\delta^{n-1}, \quad \omega_1 = (n/\Lambda_n(0))^{1/(n-1)} \sim 1$$

for any given $\delta > 1$ and n sufficiently large. Using on the analysis above, we arrive at Table 6.1 for the asymptotic behaviors for the ratios of the upper bounds over the lower bounds in the corresponding inequalities. Given that $S_{n,1}(\delta, 0)$ goes to $+\infty$ exponentially, our upper bounds and the lower bounds in Theorems 6.2, 6.2′, 6.3, and 6.3′ are very tight. These bounds, together with Theorem 2.1, lead to the qualitative behavior of $\min_{\alpha_j} \kappa_p(V)$ as α varies, depicted in Figure 1.1. From how we got the upper bounds by these inequalities, we conclude that

For a fixed α, nearly optimally conditioned V are the ones with the translated Chebyshev nodes on the symmetric interval that is slightly larger than $[-\alpha, \alpha]$ (so that $\pm\alpha$ are part of the nodes). (6.20)

In addition to Table 6.1, the analysis above yields the asymptotical speeds of $\min\limits_{\cdots} \kappa_\infty(V)$ for various cases, summarized in the following theorem.

Theorem 6.4 *We have*

$$\min_{\alpha_j} \kappa_\infty(V) = \mathcal{O}_n\left((1+\sqrt{2}\,)^n\right), \tag{6.21}$$

$$\min_{\alpha\le\delta} \kappa_\infty(V), \quad \min_{\alpha=\delta} \kappa_\infty(V) = \mathcal{O}\left((\delta^{-1}+\sqrt{1+\delta^{-2}}\,)^n\right) \quad \textit{for } \delta \le 1, \tag{6.22}$$

$$\min_{\alpha=\delta} \kappa_\infty(V) = \mathcal{O}_n\left((1+\sqrt{1+\delta^2}\,)^n\right) \quad \textit{for } \delta > 1, \tag{6.23}$$

$$\min_{\alpha\le\delta} \kappa_\infty(V) = \mathcal{O}_n\left((1+\sqrt{2}\,)^n\right) \quad \textit{for } \delta > 1. \tag{6.24}$$

Equations (6.21) – (6.24) *remain valid with V replaced by $V_{\rm sym}$.*

This is a very informative theorem, for example,

$$\min_{\alpha\le 1/2}\kappa_\infty(V),\ \min_{\alpha=1/2}\kappa_\infty(V) \;=\; \mathcal{O}\left((2+\sqrt{5}\,)^n\right), \tag{6.25}$$

$$\min_{\alpha=2}\kappa_\infty(V)=\mathcal{O}_n\left((1+\sqrt{5}\,)^n\right). \tag{6.26}$$

Except (6.22), all other equations in Theorem 6.4 are in terms of $\mathcal{O}_n(\cdots)$. It is quite natural to wonder whether this is really necessary. Examining the proof of Lemma 5.1 which results in Theorem 5.1 and (5.11), we find that the factor $1/n$ in the lower bound by (6.6) for $p=\infty$ originated from $\|V\|_\infty\ge\alpha^{n-1}$ for $\alpha\ge 1$. This could be a very crude estimate at least for those V that are (nearly) optimal. It is conceivable that in (nearly) optimal cases, nodes α_j would be well separated in a sense. Guided by Theorem 4.6, we would expect in (nearly) optimal cases $\|V\|_\infty\sim\max\{n,(\text{constant})\sqrt{n}\cdot\alpha^{n-1}\}$. If this is indeed true, the lower bounds in Theorem 6.2′ and Theorem 6.3′ can be improved to the point that they differ from the corresponding upper bounds by constant factors! We make this observation formally into the following conjecture.

Conjecture 6.1 *Let $p=\infty$. For optimally conditioned V,*

$$\sum_{j=1}^{n}|\alpha_j|^{n-1}\ge(\text{constant})^2\sqrt{n}\cdot\alpha^{n-1}, \tag{6.27}$$

and consequently all $\mathcal{O}_n(\cdots)$ in Theorem 6.4 *could be replaced by $\mathcal{O}(\cdots)$.*

Let us comment on why (6.27) would imply the second claim in this conjecture. In fact with (6.27), the lower bounds in (6.4) and (6.11) could be improved to

$$S_{n-1,1}([n/(\text{constant})]^{1/2(n-1)},0),$$

and the factor $1/n$ in the lower bound in (6.10) could be improved to $1/\sqrt{n}$. Then, asymptotical analysis similar to the one we did above would would lead to the second claim.

Our lower bound from Theorem 6.2 compares favorably to those of Gautschi and Inglese [12] which were established only for two special cases:

1. For positive nodes ($\alpha_j\ge 0$ for all j) and $n\ge 2$,

$$\begin{aligned}\kappa_\infty(V) &\ge (n-1)\left[1+\left(1-\frac{1}{n}\right)^{-1/(n-1)}\right]^{n-1}, && (6.28)\\ &= n2^{n-1}\left[1+\mathcal{O}(1/n)\right]. && (6.29)\end{aligned}$$

2. For real symmetric nodes ($\alpha_j+\alpha_{n+1-j}=0$ for all j) and $n\ge 4$,

$$\kappa_\infty(V)\ \ge\ \begin{cases}(n-2)\left[1+\left(1-\frac{2}{n}\right)^{-2/(n-1)}\right]^{(n-2)/2}, & \text{for even } n,\\ (n-3)\left[1+\left(1-\frac{3}{n}\right)^{-2/(n-1)}\right]^{(n-3)/2}, & \text{for odd } n\end{cases} \tag{6.30}$$

$$=\ \begin{cases}n(\sqrt{2})^{n-2}\left[1+\mathcal{O}(1/n)\right], & \text{for even } n\ge 4,\\ n(\sqrt{2})^{n-3}\left[1+\mathcal{O}(1/n)\right], & \text{for odd } n\ge 5.\end{cases} \tag{6.31}$$

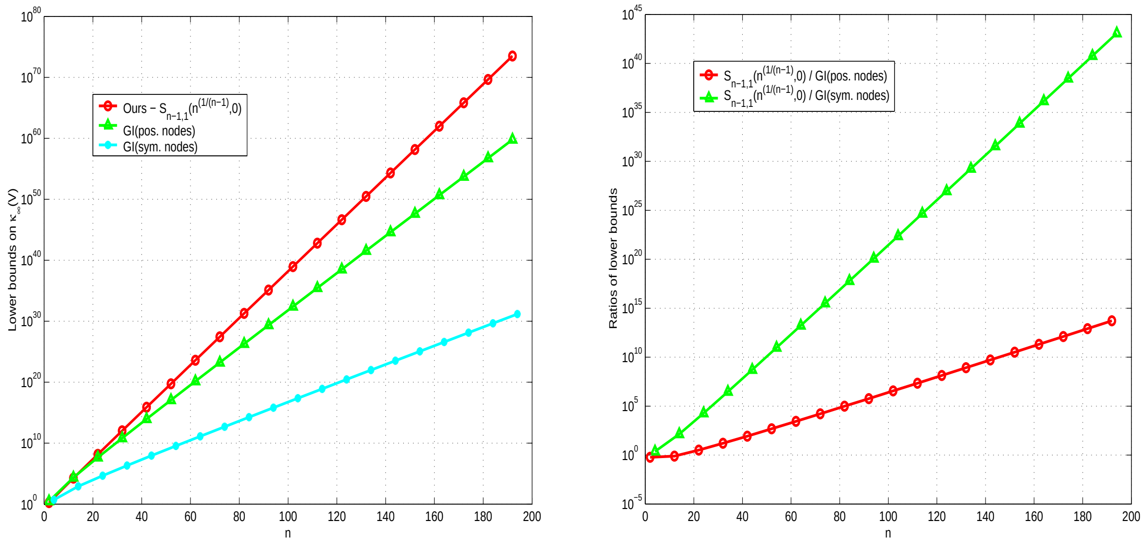


Figure 6.1: The first inequality of (6.3) for $p = \infty$ *vs.* (6.28) and (6.30)

Figure 6.1 plots our bound $S_{n-1,1}(n^{1/(n-1)}, 0)$ against these two. What shows in the figure is somewhat expected because of (6.12), (6.29), and (6.31). Later in Section 7, we shall see $\min \kappa_\infty(V)$ for positive nodes grows at a pace much faster than for general nodes, and thus the estimate (6.29) is very crude. Our lower bound is also far better than these of Tyrtyshnikov [20]:

$$\kappa_2(V) \;\geq\; 2^{n-2}/\sqrt{n} \quad \text{in general, or} \tag{6.32}$$

$$\kappa_2(V) \;\geq\; 2^{n-2} \quad \text{if all } |\alpha_j| \leq 1 \text{ or all } |\alpha_j| \geq 1. \tag{6.33}$$

Remark **6.1** What about minimizing over $[a, b]$ with $a < 0 < b$? (We leave the case $ab \geq 0$ to the next section.) We would have similar bounds if we had found a formula for $S_{n,1}(\omega, \tau)$ and had proved the conjecture we put forward at the end of Section 4. But we have not done either. For the moment, we shall explain an idea to get bounds, albeit unlikely (asymptotically) optimal, and leave the details to the reader. Let

$$\delta_{\max} = \max\{|a|, b\}, \quad \delta_{\min} = \min\{|a|, b\}.$$

Then

$$[-\delta_{\min}, \delta_{\min}] \subseteq [a, b] \subseteq [\delta_{\max}, \delta_{\max}]$$

which implies

$$\mathbb{V}_{[-\delta_{\min}, \delta_{\min}]} \subseteq \mathbb{V}_{[a,b]} \subseteq \mathbb{V}_{[-\delta_{\max}, \delta_{\max}]}$$

which implies

$$\begin{array}{ccccc} \min\limits_{\alpha \leq \delta_{\min}} \kappa_p(V) & \geq & \min\limits_{V \in \mathbb{V}_{[a,b]}} \kappa_p(V) & \geq & \min\limits_{\alpha \leq \delta_{\max}} \kappa_p(V) \\ \mathrel{\rotatebox{90}{$\geq$}} & & & & \mathrel{\rotatebox{90}{$\geq$}} \\ \min\limits_{\alpha \leq \delta_{\min}} \kappa_p(V_{\text{sym}}) & & \geq & & \min\limits_{\alpha \leq \delta_{\max}} \kappa_p(V_{\text{sym}}) \end{array} \tag{6.34}$$

Depending on the magnitudes of $\delta_{\min}$ and $\delta_{\max}$, various inequalities on $\min_{V\in\mathbb{V}_{[a,b]}} \kappa_p(V)$ can be established with the help of Theorems 6.3 and 6.3′.

7 Condition numbers of Vandermonde matrices whose nodes are in $[a,b]$ with $0\le a<b$

Assume throughout this section

$$0\le a<b.$$

Notice that the case $a<b\le 0$ can be turned into this case by reversing the signs of all α_j while leaving norms $\|V\|_p$ and $\|V^{-1}\|_p$ unchanged. So results in what follows apply to the case $a<b\le 0$ as well after minor modifications.

Let $0\le a\le \alpha_j\le \max_j\alpha_j=\alpha\le b$. Then $\tau\le -1$, and thus by Theorem 2.1 $S_{n-1,p'}(\omega,\tau)$ is decreasing in ω and increasing in $|\tau|$. Given α_j and thus the constraints $0\le a\le\alpha_j\le b$ on a and b, it follows from (4.3) that ω achieves its smallest value and $|\tau|$ its biggest values at the same time when

$$a=\min_j\alpha_j,\quad b=\alpha=\max_j\alpha_j \tag{7.1}$$

at which the right-hand side of (5.9) is maximized over all possible feasible a and b subject to $0\le a\le\alpha_j\le b$. For this reason assignments (7.1) will be kept throughout this section, unless otherwise explicitly stated.

Theorem 7.1 *Suppose $\alpha_j\ge 0$, and a and b defined by* (7.1). *Then*

$$\begin{aligned}\kappa_p(V) &\ge \max\left\{S_{n-1,p'}(\omega,\tau),\frac{\alpha^{n-1}S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}\right\}, &(7.2)\\ &\ge \max\left\{S_{n-1,p'}(\alpha/2,1),\frac{\alpha^{n-1}S_{n-1,p'}(\alpha/2,1)}{n^{1/p'}}\right\}, &(7.3)\\ &= \begin{cases}S_{n-1,p'}(\alpha/2,1), & \text{if } \alpha\le n^{1/[p'(n-1)]},\\ \dfrac{\alpha^{n-1}S_{n-1,p'}(\alpha/2,1)}{n^{1/p'}}, & \text{if } \alpha> n^{1/[p'(n-1)]}.\end{cases} &(7.4)\end{aligned}$$

Proof: (7.2) is (5.9) with (7.1). (7.3) holds because of $\omega\le\alpha/2$ and $\tau\le -1$, and the monotonicity of $S_{n-1,p'}(\omega,\tau)$ in its two arguments by Theorem 2.1. Also by Theorem 2.1, the first quantity within $\max\{\cdots\}$ in (7.3) is decreasing; while the second one is increasing. Therefore the right-hand side of (7.3) achieves its minimum when the two equal, i.e.,

$$n^{1/p'}=\alpha^{n-1}\quad\Rightarrow\quad \alpha=n^{1/[p'(n-1)]}$$

which yields (7.4). ∎

Theorem 7.2

$$S_{n-1,p'}\left(n^{1/[p'(n-1)]}\Big/2,1\right)\le \min_{\alpha_j\ge 0}\kappa_p(V)\le n^{1/p}\frac{\sqrt{2}}{4}S_{n,1}(1/2,1). \tag{7.5}$$

Proof: The first inequality is true because the right-hand side of (7.3) achieves its minimum at $\alpha = n^{1/[p'(n-1)]}$. We now prove the second one. To this end, take $0 = a < b = 1$, and consider V with translated Chebyshev nodes x_j. Then $\|V\|_1 \le n$ and $\|V\|_\infty = n$, and thus by (3.7)

$$\|V\|_p \le \|V\|_1^{1/p}\|V\|_\infty^{1/p'} \le n.$$

Apply Theorem 4.5 to the case $0 = a < b = 1$ (and thus $\omega = 1/2$ and $\tau = -1$) to get $\|V^{-1}\|_\infty \le (2\sqrt{2}\,n)^{-1}\, S_{n,1}(1/2, 1)$ and then to get

$$\|V^{-1}\|_p \le n^{1/p}\|V^{-1}\|_\infty \le n^{1/p}(2\sqrt{2}\,n)^{-1}\, S_{n,1}(1/2, 1).$$

So for this V, $\kappa_p(V) \le n^{1/p}(2\sqrt{2})^{-1}\, S_{n,1}(1/2, 1)$, as needed. ∎

The second inequality in (7.5) was proved by simply picking a special V with the translated Chebyshev nodes in $[0, 1]$. For the same reason that led to Theorem 6.2′, for $p = \infty$ a tighter upper bound is possible by using the V with the translated Chebyshev nodes in $[0, b^+_{\rm opt}]$, where $b^+_{\rm opt} = 2\,(n/\Lambda_n(1))^{1/(n-1)}$ as in Theorem 4.6. This gives the following theorem.

Theorem 7.2′

$$S_{n-1,1}\left(n^{1/(n-1)}\Big/ 2, 1\right) \le \min_{\alpha_j\ge 0} \kappa_\infty(V) \le \frac{b^+_{\rm opt}}{2\sqrt{1+b^+_{\rm opt}}}\, S_{n,1}(b^+_{\rm opt}/2, 1), \tag{7.6}$$

where $b^+_{\rm opt} = 2\,(n/\Lambda_n(1))^{1/(n-1)}$.

Proof: It is a consequence of Theorem 7.2 for $p = \infty$ and Theorem 4.6. ∎

Theorem 7.3 *Given* $0 \le \gamma < \delta$, *let*

$$\omega_0 = \frac{\delta-\gamma}{2}, \quad \omega' = \frac{2}{1+c}\omega_0, \quad \tau_0 = -\frac{\delta+\gamma}{\delta-\gamma}, \quad \tau' = \tau_0\frac{1+c}{2}\left(1 - \frac{1}{\tau_0}\frac{1-c}{1+c}\right),$$

where $c = \cos\frac{\pi}{2n}$. *If* $\delta < 1$, *then*

$$S_{n-1,p'}(\omega_0, \tau_0) \le \min_{\substack{V\in\mathbb{V}_{[\gamma,\delta]}\\ \alpha=\delta}} \kappa_p(V) \le n^{1/p}\,\Xi\, S_{n,1}(\omega', \tau') \le n^{1/p}\,\Xi\, S_{n,1}(\omega_0, \tau_0), \tag{7.7}$$

where

$$\begin{aligned}\Xi &= \frac{\delta-\gamma}{\sqrt{1+c}\sqrt{1+\gamma}\sqrt{1+c+2\delta-\gamma+\gamma c}} \qquad (7.8)\\ &= \frac{\delta-\gamma}{2\sqrt{1+\gamma}\sqrt{1+\delta}} + \frac{(\delta-\gamma)(\delta+\gamma+2)}{16\sqrt{1+\gamma}(1+\delta)^{3/2}}\frac{\pi^2}{4n^2} + \mathcal{O}\left(\frac{1}{n^4}\right);\end{aligned}$$

and if $\delta > 1$, *then*

$$\frac{\delta^{n-1}S_{n-1,p'}(\omega_0, \tau_0)}{n^{1/p'}} \le \min_{\substack{V\in\mathbb{V}_{[\gamma,\delta]}\\ \alpha=\delta}} \kappa_p(V) \le n^{1/p}\,\Xi\delta^{n-1}\, S_{n,1}(\omega', \tau') \le n^{1/p}\,\Xi\delta^{n-1}\, S_{n,1}(\omega_0, \tau_0). \tag{7.9}$$

Proof: Pick any $0 \le a \le \gamma < \delta \le b$, we still have (5.11). For $a = \gamma$ and $b = \delta$, in particular, we deduce the first inequalities in (7.7) and (7.9). To prove the second inequalities, we take $a = \gamma$, and let b be determined. Set $\alpha_j = x_j$, and then pick b such that $x_1 = \delta$, i.e.,

$$\delta = x_1 = \omega(t_1 - \tau) = \frac{b-a}{2} \cos \frac{\pi}{2n} + \frac{b+a}{2} = b\frac{1+c}{2} + a\frac{1-c}{2}$$

which gives

$$b = \delta + (\delta - \gamma)\frac{1-c}{1+c}, \quad \omega = \omega_0 \frac{2}{1+c} = \omega', \quad \tau = \tau_0 \frac{1+c}{2}\left(1 - \frac{1}{\tau_0}\frac{1-c}{1+c}\right) = \tau'.$$

Apply Theorem 4.5 to get

$$\|V^{-1}\|_\infty \le \frac{\Xi}{n} S_{n,1}(\omega, \tau).$$

Now $\|V\|_p \le n$ if $\delta \le 1$ and $\|V\|_p \le n\delta^{n-1}$ if $\delta \ge 1$ and $\|V^{-1}\|_p \le n^{1/p}\|V^{-1}\|_\infty$. Thus if $\delta \le 1$,

$$\kappa_p(V) \le n^{1+1/p}\|V^{-1}\|_\infty \le n^{1/p}\Xi\, S_{n,1}(\omega, \tau);$$

and if $\delta \ge 1$,

$$\kappa_p(V) \le n^{1+1/p}\delta^{n-1}\|V^{-1}\|_\infty \le n^{1/p}\delta^{n-1}\,\Xi\, S_{n,1}(\omega, \tau).$$

This completes the proofs for the second inequalities in (7.7) and (7.9). The third inequalities hold because $\omega' > \omega_0$ and $|\tau'| \le |\tau_0|$. ∎

The theorem below is essentially Theorem 7.3 for $\gamma = 0$.

Theorem 7.4 *Let $\delta > 0$, and let $\delta' = [2/(1+c)]\delta \ge \delta$, where $c = \cos \frac{\pi}{2k}$. If $\delta < 1$, then*

$$S_{n-1,p'}(\delta/2, 1) \le \min_{V \in \mathbb{V}_{[0,\delta]}} \kappa_p(V) \le \min_{\substack{V \in \mathbb{V}_{[0,\delta]} \\ \alpha = \delta}} \kappa_p(V) \le n^{1/p} \frac{\delta'}{2\sqrt{1+\delta'}} S_{n,1}(\delta'/2, 1); \quad (7.10)$$

and if $\delta > 1$, then

$$\frac{\delta^{n-1} S_{n-1,p'}(\delta/2, 1)}{n^{1/p'}} \le \min_{\substack{V \in \mathbb{V}_{[0,\delta]} \\ \alpha = \delta}} \kappa_p(V) \le n^{1/p}\, \frac{1}{2}\left(\frac{1+c}{2}\right)^{n-1} \frac{(\delta')^n}{\sqrt{1+\delta'}}\, S_{n,1}(\delta'/2, 1), \quad (7.11)$$

$$S_{n-1,p'}\left(n^{1/[p'(n-1)]}/2, 1\right) \le \min_{V \in \mathbb{V}_{[0,\delta]}} \kappa_p(V) \le n^{1/p}\, \frac{\sqrt{2}}{4}\, S_{n,1}(1/2, 1). \quad (7.12)$$

Proof: 1) Observe that $\{V \in \mathbb{V}_{[0,\delta]} \,:\, \alpha = \delta\} \subset \mathbb{V}_{[0,\delta]}$. So

$$\min_{V \in \mathbb{V}_{[0,\delta]}} \kappa_p(V) \le \min_{\substack{V \in \mathbb{V}_{[0,\delta]} \\ \alpha = \delta}} \kappa_p(V). \quad (7.13)$$

This gives the middle inequality in (7.10).

2) Apply Theorem 7.3 with $0 = \gamma < 1 \le \delta$ to obtain the last inequality in (7.10) and to obtain (7.11).

3) It follows from (7.4) that for $0 \le \alpha \le \delta \le 1$,

$$\kappa_p(V) \ge S_{n-1,p'}(\alpha/2,1) \ge S_{n-1,p'}(\delta/2,1)$$

by Theorem 2.1. So the first inequality in (7.10) holds.

4) A proof of (7.12) can be done in the same way as for Theorem 7.2. ∎

There are stronger versions of (7.11) and (7.12) for $p = \infty$, too.

Theorem 7.4′ *Let $\delta > 1$, and let $\delta' = [2/(1+c)]\delta \ge \delta$, where $c = \cos\frac{\pi}{2k}$. Then*

$$\frac{\delta^{n-1} S_{n-1,1}(\delta/2,1)}{n} \le \min_{\substack{V\in\mathbb{V}_{[0,\delta]}\\ \alpha=\delta}} \kappa_\infty(V) \le \frac{\max\{n, 2^{-(n-1)}\Lambda_n(1)(\delta')^{n-1}\}}{n} \frac{\delta'}{2\sqrt{1+\delta'}} S_{n,1}(\delta'/2,1), \tag{7.14}$$

$$S_{n-1,1}\left(n^{1/(n-1)}/2,1\right) \le \min_{V\in\mathbb{V}_{[0,\delta]}} \kappa_\infty(V) \le \frac{\delta_1}{2\sqrt{1+\delta_1}} S_{n,1}(\delta_1/2,1), \tag{7.15}$$

where $\delta_1 = \min\{\delta', 2\,(n/\Lambda_n(1))^{1/(n-1)}\}$. It can be seen that $\delta_1 = 2\,(n/\Lambda_n(1))^{1/(n-1)}$ for n sufficiently large.

Proof: Only the second inequalities in (7.14) and (7.15) need proofs. For (7.14), it follows from the proof of Theorem 7.3 for $\gamma = 0$, where instead of $\|V\|_\infty \le n\delta^{n-1}$ we use $\|V\|_\infty = \max\{n, \omega^{n-1}\Lambda_n(1)\}$, which is proportional to $\sqrt{n}\,\delta^{n-1}$. The second inequality in (7.15) is obtained by approximately minimizing

$$\frac{\max\{n, 2^{-(n-1)}\Lambda_n(1)(\alpha')^{n-1}\}}{n} \frac{\alpha'}{2\sqrt{1+\alpha'}} S_{n,1}(\alpha'/2,1)$$

subject to $\alpha \le \delta$, where $\alpha' = [2/(1+c)]\alpha$. That $\delta_1 = 2\,(n/\Lambda_n(1))^{1/(n-1)}$ for n sufficiently large is due to $(n/\Lambda_n(1))^{1/(n-1)} \sim 2^{-1}\,(n\pi)^{1/[2(n-1)]} \sim 2^{-1}$. ∎

We shall now investigate the tightness of the upper bounds and the lower bounds in Theorems 7.2, 7.2′, 7.3, 7.4, and 7.4′. Again, we shall do so only for $p = \infty$ (and thus $p' = 1$). Since

$$1 + 2n^{-1/(n-1)} = 3 - 2\frac{\ln(n-1)}{n-1} + \frac{-2+\ln^2(n-1)}{(n-1)^2} + \cdots,$$

Theorem 7.2 says that $\kappa_\infty(V)$ is no better than

$$\begin{aligned} T_{n-1}(1+2n^{-1/(n-1)}) &= \frac{(n-1)^{-1/\sqrt{2}}}{2}(3+2\sqrt{2})^{n-1}\left[1+\mathcal{O}\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\ &\quad + \frac{(n-1)^{1/\sqrt{2}}}{2}(3+2\sqrt{2})^{1-n}\left[1-\mathcal{O}\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\ &\sim \frac{(3+2\sqrt{2})^{n-1}}{2(n-1)^{1/\sqrt{2}}}. \end{aligned} \tag{7.16}$$

On the other hand, $S_{n,1}(1/2,1) = \frac{1}{2}(3+2\sqrt{2}\,)^n + \frac{1}{2}(3+2\sqrt{2}\,)^{-n}$, and for $b^+_{\rm opt} = 2\,(n/\Lambda_n(1))^{1/(n-1)}$,

$$\begin{aligned} S_{n,1}(b^+_{\rm opt}/2,1) &= \frac{(n\pi)^{-\sqrt{2}/4}}{2}(3+2\sqrt{2})^n\left[1+\mathcal{O}\left(\frac{\ln(n-1)}{n-1}\right)\right] \\ &\quad + \frac{(n\pi)^{\sqrt{2}/4}}{2}(3+2\sqrt{2})^{-n}\left[1-\mathcal{O}\left(\frac{\ln(n-1)}{n-1}\right)\right] \\ &\sim \frac{(3+2\sqrt{2})^n}{2(n\pi)^{\sqrt{2}/4}}. \end{aligned} \tag{7.17}$$

Therefore

$$\begin{aligned} \frac{S_{n,1}(1/2,1)}{S_{n-1,1}\left(n^{1/(n-1)}/2,1\right)} &\sim (3+2\sqrt{2})\,n^{1/\sqrt{2}}, \\ \frac{S_{n,1}(b^+_{\rm opt}/2,1)}{S_{n-1,1}\left(n^{1/(n-1)}/2,1\right)} &\sim (3+2\sqrt{2})\,(n\pi)^{\sqrt{2}/4}, \\ \min_{\alpha_j\ge 0}\kappa_\infty(V) &= \mathcal{O}_n\left((3+2\sqrt{2}\,)^n\right). \end{aligned} \tag{7.18}$$

Even though $b^+_{\rm opt}\sim 1$, $S_{n,1}(b^+_{\rm opt}/2,1)$ is about $(\pi/n)^{\sqrt{2}/4}$ times smaller than $S_{n,1}(1/2,1)$. Now it is clear that (6.28), the estimate in [12], is far from sharp. Let us turn to Theorem 7.3. First, we claim

$$\phi \stackrel{\rm def}{=} \left(\frac{\frac{1}{\omega'}+|\tau'|+\sqrt{\left(\frac{1}{\omega'}+|\tau'|\right)^2-1}}{\frac{1}{\omega_0}+|\tau_0|+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\right)^2-1}}\right)^n \sim 1. \tag{7.19}$$

To this end, we notice that $\zeta \stackrel{\rm def}{=} \left(1-\frac{1}{\tau_0}\frac{1-c}{1+c}\right)\ge 1$, and $\frac{1+c}{2}\zeta\le 1$, and

$$\begin{aligned} \frac{1}{\omega'}+|\tau'|+\sqrt{\left(\frac{1}{\omega'}+|\tau'|\right)^2-1} &= \frac{1+c}{2}\left(\frac{1}{\omega_0}+|\tau_0|\zeta+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\zeta\right)^2-\left(\frac{2}{1+c}\right)^2}\right) \\ &\ge \frac{1+c}{2}\left(\frac{1}{\omega_0}+|\tau_0|+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\right)^2-1}\right). \end{aligned}$$

Therefore $1\ge\phi\ge(\frac{1+c}{2})^n\sim 1$ because $n\ln\frac{1+c}{2}\sim-\frac{\pi^2}{16n}\to 0$. This proves (7.19). With (7.19), we have, by (2.17),

$$\frac{S_{n,1}(\omega',\tau')}{S_{n-1,1}(\omega_0,\tau_0)}\sim\frac{S_{n,1}(\omega_0,\tau_0)}{S_{n-1,1}(\omega_0,\tau_0)}\sim\left(\frac{1}{\omega_0}+|\tau_0|+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\right)^2-1}\right). \tag{7.20}$$

Inequality	Ratio $\sim$	for bounds on
(7.5)	$\frac{4+3\sqrt{2}}{4} \times n^{1/\sqrt{2}}$	$\min_{\alpha_j \geq 0}$
(7.6)	$\frac{\sqrt{2}}{4}\pi^{\sqrt{2}/4} \times n^{\sqrt{2}/4}$	
(7.7)	$\frac{(\sqrt{1+\gamma}+\sqrt{1+\delta})^2}{2\sqrt{1+\gamma}\,\sqrt{1+\delta}} \times n^0$	$\min\limits_{V\in\mathbb{V}_{[\gamma,\delta]}}$ subject to $\alpha=\delta$ for $\delta\leq 1$
(7.9)	$\frac{(\sqrt{1+\gamma}+\sqrt{1+\delta})^2}{2\sqrt{1+\gamma}\,\sqrt{1+\delta}} \times n^1$	$\min\limits_{V\in\mathbb{V}_{[\gamma,\delta]}}$ subject to $\alpha=\delta$ for $\delta > 1$
(7.10)	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}} \times n^0$	$\min\limits_{V\in\mathbb{V}_{[0,\delta]}}$ subject to $\alpha=\delta$ for $\delta\leq 1$
(7.11)	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}} \times n^1$	$\min\limits_{V\in\mathbb{V}_{[0,\delta]}}$ subject to $\alpha=\delta$ for $\delta > 1$
(7.14)	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}}\frac{1}{\pi} \times n^{1/2}$	
(7.12)	$\frac{4+3\sqrt{2}}{4} \times n^{1/\sqrt{2}}$	$\min\limits_{V\in\mathbb{V}_{[0,\delta]}}$ subject to $\alpha\leq\delta$ for $\delta > 1$
(7.15)	$\frac{\sqrt{2}}{4}\pi^{\sqrt{2}/4} \times n^{\sqrt{2}/4}$	

Table 7.1: Ratios of the upper bounds over the lower bounds for $p=\infty$ and nonnegative nodes

For the bounds in Theorem 7.4′, we notice

$$\begin{aligned}\max\{n, 2^{-(n-1)}\Lambda_n(1)(\delta')^{n-1}\} &= 2^{-(n-1)}\Lambda_n(1)(\delta')^{n-1} \sim \sqrt{\frac{n}{\pi}}\,\delta^{n-1}, \\ \delta_1 &= 2\,(n/\Lambda_n(1))^{1/(n-1)} \sim 1.\end{aligned}$$

for any given $\delta > 1$ and n sufficiently large. Using on the analysis above, we arrive at Table 7.1 for the asymptotic behaviors for the ratios of the upper bounds over the lower bounds in the corresponding inequalities. The conclusions are that these bounds are very tight. These bounds, together with Theorem 2.1, lead to the qualitative behavior of $\min_{\alpha_j\geq 0}\kappa_\infty(V)$ as α varies, depicted in Figure 1.1. Also,

For a fixed α, nearly optimally conditioned V are the ones with the translated Chebyshev nodes on an interval slightly larger than $[0,\alpha]$ (so that α is a node).	(7.21)

In addition to Table 7.1, the analysis above yields the asymptotical speeds of $\min\limits_{\cdots}\kappa_\infty(V)$ for various cases, summarized in the following theorem.

Theorem 7.5 *For $0\leq\gamma<\delta$,*

$$\min_{\substack{V\in\mathbb{V}_{[\gamma,\delta]}\\ \alpha=\delta}} \kappa_\infty(V) = \mathcal{O}\left(\left[\frac{1}{\omega_0}+|\tau_0|+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\right)^2-1}\,\right]^n\right) \qquad \textit{for } \delta\leq 1, \quad (7.22)$$

$$\min_{\substack{V\in\mathbb{V}_{[\gamma,\delta]}\\ \alpha=\delta}} \kappa_\infty(V) = \mathcal{O}_n\left(\delta^n\left[\frac{1}{\omega_0}+|\tau_0|+\sqrt{\left(\frac{1}{\omega_0}+|\tau_0|\right)^2-1}\,\right]^n\right) \quad \textit{for } \delta>1; \quad (7.23)$$

$$\min_{V\in\mathbb{V}_{[0,\delta]}} \kappa_\infty(V),\quad \min_{\substack{V\in\mathbb{V}_{[0,\delta]}\\ \alpha=\delta}} \kappa_\infty(V) = \mathcal{O}\left(\left[\delta^{-1/2}+1+\sqrt{1+\delta^{-1}}\,\right]^{2n}\right) \quad \textit{for } \delta\le 1, \quad (7.24)$$

$$\min_{\substack{V\in\mathbb{V}_{[0,\delta]}\\ \alpha=\delta}} \kappa_\infty(V) = \mathcal{O}_n\left((1+\sqrt{1+\delta}\,)^{2n}\right) \quad \textit{for } \delta>1, \quad (7.25)$$

$$\min_{V\in\mathbb{V}_{[0,\delta]}} \kappa_\infty(V) = \mathcal{O}_n\left((3+2\sqrt{2}\,)^{n}\right) \quad \textit{for } \delta>1; \quad (7.26)$$

And finally,

$$\min_{\alpha_j\ge 0} \kappa_\infty(V) = \mathcal{O}_n\left((3+2\sqrt{2}\,)^{n}\right). \quad (7.27)$$

With constants and modest factors n^d hidden, this theorem is perhaps more informative than the previous ones, for example,

$$\min_{\substack{V\in\mathbb{V}_{[1/4,1/2]}\\ \alpha=1/2}} \kappa_\infty(V) = \mathcal{O}\left((11+2\sqrt{30}\,)^{n}\right),$$

$$\min_{\substack{V\in\mathbb{V}_{[1,2]}\\ \alpha=2}} \kappa_\infty(V) = \mathcal{O}_n\left((10+4\sqrt{6}\,)^{n}\right),$$

$$\min_{V\in\mathbb{V}_{[0,1/2]}} \kappa_\infty(V),\quad \min_{\substack{V\in\mathbb{V}_{[0,1/2]}\\ \alpha=1/2}} \kappa_\infty(V) = \mathcal{O}\left((5+2\sqrt{6}\,)^{n}\right),$$

$$\min_{\substack{V\in\mathbb{V}_{[0,2]}\\ \alpha=2}} \kappa_\infty(V) = \mathcal{O}_n\left((4+2\sqrt{3}\,)^{n}\right).$$

For the same reason that led to Conjecture 6.1, we have

Conjecture 7.1 *Let $p=\infty$, and all $\alpha_j\ge 0$. For optimally conditioned V,*

$$\sum_{j=1}^{n} |\alpha_j|^{n-1} \ge (\text{constant})^2\sqrt{n}\cdot\alpha^{n-1}, \quad (7.28)$$

and consequently all $\mathcal{O}_n(\cdots)$ in (7.25), (7.26), and (7.27) could be replaced by $\mathcal{O}(\cdots)$. The $\mathcal{O}_n(\cdots)$ in (7.23) could be replaced by $\mathcal{O}(\cdots)$, too.

8 Rectangular Vandermonde matrices – a general theorem

The results we have established so far can be extended to a $k \times n$ rectangular Vandermonde matrix by which we mean the first k rows of V in (1.1), denoted by V_k.

V_k is rectangular, unless $k = n$; and it has rank k if there are at least k distinct nodes among $\{\alpha_j\}_{j=1}^n$. Then $V_k^T u = 0$ for some $u \in \mathbb{R}^k$ if and only if $u = 0$.

Our first task here is to explain what we mean by a p-condition number $\kappa_p(V_k)$ of V_k. Recall that $\kappa_p(V) \stackrel{\text{def}}{=} \|V\|_p \|V^{-1}\|_p$. It is natural to replace $\|V\|_p$ by $\|V_k\|_p$ which is still well-defined. But what about extending $\|V^{-1}\|_p$? For any given $1 \le p \le +\infty$, we define

$$\mathsf{lub}_p(V_k) \stackrel{\text{def}}{=} \min_{u \neq 0} \frac{\|V_k^T u\|_{p'}}{\|u\|_{p'}}. \tag{8.1}$$

Such definition is unlikely new, and is consistent with the square matrix case. In fact for $k = n$, it can be shown that $\mathsf{lub}_p(V) = \|V^{-1}\|_p^{-1}$. Therefore, we define

$$\kappa_p(V_k) \stackrel{\text{def}}{=} \|V_k\|_p / \mathsf{lub}_p(V_k). \tag{8.2}$$

Lemma 8.1

$$\max\{k, k\alpha^{k-1}\} \ge \|V_k\|_1 = \sum_{j=1}^{k} |\alpha|^{j-1} \ge \max\{1, \alpha^{k-1}\}, \tag{8.3}$$

$$\max\{n, n\alpha^{k-1}\} \ge \|V_k\|_\infty = \max\left\{ n, \sum_{j=1}^{n} |\alpha_j|^{k-1} \right\} \ge \max\{n, \alpha^{k-1}\}, \tag{8.4}$$

$$k^{1/p} n^{1/p'} \max\{1, \alpha^{k-1}\} \ge \|V_k\|_p \ge \max\{n^{1/p'}, \alpha^{k-1}\}, \tag{8.5}$$

$$\mathsf{lub}_p(V_k) \le \frac{n^{1/p'}}{S_{k-1,p'}(\omega,\tau)} \le \frac{k^{1/p} n^{1/p'}}{S_{k-1,1}(\omega,\tau)}. \tag{8.6}$$

An even sharper bound than the second inequality in (8.6) can be proved too, similarly to (5.6). If $-a = b$, then

$$\mathsf{lub}_p(V_k) \le \left(\frac{\lceil k/2 \rceil}{k} \right)^{1/p} \cdot \frac{k^{1/p} n^{1/p'}}{S_{k-1,1}(\omega, 0)}, \tag{8.7}$$

As a result of Lemma 8.1, we have, similarly to theorems in Section 5,

Theorem 8.1

$$\kappa_p(V_k) \quad \ge \quad \max\left\{ S_{k-1,p'}(\omega,\tau), \frac{\alpha^{k-1} S_{k-1,p'}(\omega,\tau)}{n^{1/p'}} \right\}, \tag{8.8}$$

$$\kappa_p(V_k) \quad \ge \quad \begin{cases} S_{k-1,p'}(\alpha, 0), & \text{if } \alpha \le n^{1/[p'(k-1)]}, \\ \dfrac{\alpha^{k-1} S_{k-1,p'}(\alpha,0)}{n^{1/p'}}, & \text{if } \alpha > n^{1/[p'(k-1)]}. \end{cases} \tag{8.9}$$

Proof: (8.8) is an immediate consequence of Lemma 8.1. Now apply (8.8) with $-a = b = \alpha$ (and thus $\omega = \alpha$ and $\tau = 0$) to get

$$\kappa_p(V_k) \geq \max\left\{S_{k-1,p'}(\alpha, 0), \frac{\alpha^{k-1}S_{k-1,p'}(\alpha,0)}{n^{1/p'}}\right\}. \tag{8.10}$$

By Theorem 2.1., the first quantity within $\max\{\cdots\}$ in (8.10) is decreasing in α; while the second one is increasing in α. Therefore the right-hand side of (8.10) achieves its minimum when the two equal, i.e.,

$$n^{1/p'} = \alpha^{k-1} \quad \Rightarrow \quad \alpha = n^{1/[p'(k-1)]}$$

which yields (8.9). ∎

Lemma 8.2 *Let $Z \in \mathbb{R}^{k\times n}$ with rank k. For $1 \leq p,\ q \leq \infty$, we have*

$$\min\{k^{1/p-1/q}, n^{1/q-1/p}\}\mathsf{lub}_q(Z) \leq \mathsf{lub}_p(Z) \leq \max\{k^{1/p-1/q}, n^{1/q-1/p}\}\mathsf{lub}_q(Z). \tag{8.11}$$

Proof: Similarly to (3.4) and (3.5), we have for $u \in \mathbb{R}^k$

$$\begin{aligned}
n^{-(1/q'-1/p')}\frac{\|Z^Tu\|_{q'}}{\|u\|_{q'}} \leq \frac{\|Z^Tu\|_{p'}}{\|u\|_{p'}} &\leq k^{1/q'-1/p'}\frac{\|Z^Tu\|_{q'}}{\|u\|_{q'}} \quad \text{for } p' \geq q',\\
k^{-(1/p'-1/q')}\frac{\|Z^Tu\|_{q'}}{\|u\|_{q'}} \leq \frac{\|Z^Tu\|_{p'}}{\|u\|_{p'}} &\leq n^{1/p'-1/q'}\frac{\|Z^Tu\|_{q'}}{\|u\|_{q'}} \quad \text{for } p' < q'.
\end{aligned}$$

This, together with the definition (8.1) of lub_p, lead to (8.11). ∎

Lemma 8.3 *Let $W \in \mathbb{R}^{k\times k}$ with rank k, and let*

$$Z = (\underbrace{W\ W\ \cdots\ W}_{\lfloor n/k\rfloor}\ W(:, 1:\ell)) \in \mathbb{R}^{k\times n}, \tag{8.12}$$

where $\ell = n - k\lfloor n/k\rfloor$, and $W(:, 1:\ell)$ is the first ℓ columns of W in the MATLAB-like notation. Then

$$\begin{aligned}
\mathsf{lub}_p(Z) &\leq \phi_p(k,n)\lceil n/k\rceil\|W^{-1}\|_2^{-1} \leq \phi_p(k,n)\lceil n/k\rceil\sqrt{k}\|W^{-1}\|_\infty^{-1}, &(8.13)\\
\mathsf{lub}_p(Z) &\geq [\psi_p(k,n)]^{-1}\lfloor n/k\rfloor\|W^{-1}\|_2^{-1} \geq \left[\psi_p(k,n)\sqrt{k}\right]^{-1}\lfloor n/k\rfloor\|W^{-1}\|_\infty^{-1}, &(8.14)
\end{aligned}$$

where

$$\phi_p(k,n) \stackrel{\text{def}}{=} \max\{k^{1/p-1/2}, n^{1/2-1/p}\}, \quad \psi_p(k,n) \stackrel{\text{def}}{=} \max\{n^{1/p-1/2}, k^{1/2-1/p}\}. \tag{8.15}$$

As a consequence,

$$\frac{[\phi_p(k,n)]^{-1}}{\sqrt{k}\lceil n/k\rceil} \leq \frac{\mathsf{lub}_p(Z)^{-1}}{\|W^{-1}\|_\infty} \leq \frac{\psi_p(k,n)\sqrt{k}}{\lfloor n/k\rfloor}. \tag{8.16}$$

Proof: Notation $X \le Y$ for two symmetric matrices means that $Y - X$ is nonnegative definite. We have

$$\lfloor n/k \rfloor W W^T \le Z Z^T \le \lceil n/k \rceil W W^T. \tag{8.17}$$

Therefore

$$\lfloor n/k \rfloor \|W^{-1}\|_2^{-1} \le \mathsf{lub}_2(Z) = \sqrt{\min_u \frac{u^T Z Z^T u}{u^T u}} \le \lceil n/k \rceil \|W^{-1}\|_2^{-1}.$$

As a result, we have (8.13) and (8.14) by Lemma 8.2 with $q = 2$. ∎

9 Rectangular Vandermonde matrices whose nodes are in $[a, b]$ with $-a = b$

Lemma 9.1 *In* Lemma 8.3, *let $W \in \mathbb{R}^{k\times k}$ be the Vandermonde matrix with nodes being the zeros of $T_k(x; \omega, 0)$ defined with $-a = b = \delta > 0$, and define $V_k = Z$ there. Then*

$$\delta \min\left\{1, \frac{1+\delta}{1+\delta^2}\right\} \frac{[\phi_p(k,n)]^{-1}}{k\sqrt{k}\,\lceil n/k \rceil} \le \frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\delta, 0)} \le \delta \max\left\{1, \frac{1+\delta}{1+\delta^2}\right\} \frac{3^{3/4}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k \rfloor}, \tag{9.1}$$

where $\phi_p(k, n)$ and $\psi_p(k, n)$ are defined by (8.15).

Proof: Apply Theorem 4.3 to W here which is a $k \times k$ Vandermonde matrices with nodes being zeros of $T_k(x; \omega, 0)$, to get

$$\delta \min\left\{1, \frac{1+\delta}{1+\delta^2}\right\} \frac{1}{k} \le \frac{\|W^{-1}\|_\infty}{S_{k,1}(\delta, 0)} \le \delta \max\left\{1, \frac{1+\delta}{1+\delta^2}\right\} \frac{3^{3/4}}{2k}. \tag{9.2}$$

Now apply (8.16), noticing

$$\frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\delta, 0)} = \frac{\mathsf{lub}_p(V_k)^{-1}}{\|W^{-1}\|_\infty} \cdot \frac{\|W^{-1}\|_\infty}{S_{k,1}(\delta, 0)},$$

to complete the proof. ∎

Theorem 9.1

$$S_{k-1,p'}(n^{1/[p'(k-1)]}, 0) \le \min_{\alpha_j} \kappa_p(V_k) \le \frac{(\sqrt{2}+1)3^{3/4}}{4} k^{1/p} n^{(2k-1)/[p'(k-1)]} \times \frac{\psi_p(k,n)}{\sqrt{k}\lfloor n/k \rfloor} S_{k,1}(n^{1/[p'(k-1)]}, 0). \tag{9.3}$$

Proof: The first inequality holds because the right-hand side of (8.10) achieves its minimum at $\alpha = n^{1/[p'(k-1)]}$. We now prove the second one. To this end, we shall use the V_k constructed in Lemma 9.1 with $\delta = n^{1/[p'(k-1)]} > 1$. We have by (9.1) that

$$\mathsf{lub}_\infty(V_k)^{-1} \le n^{1/[p'(k-1)]} \frac{\sqrt{2}+1}{2} \frac{3^{3/4}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k \rfloor} S_{k,1}(n^{1/[p'(k-1)]}, 0)$$

which, together with $\|V_k\|_\infty \le k^{1/p} n^{1/p'} \delta^{k-1} = k^{1/p} n^{2/p'}$ by (8.5), completes the proof. ∎

Theorem 9.2 *Let $\delta > 0$, and set $\delta' = \frac{\delta}{\cos\frac{\pi}{2n}}$. If $\delta \le n^{1/[p'(k-1)]}$, then*

$$S_{k-1,p'}(\delta, 0) \le \min_{\alpha\le\delta} \kappa_p(V_k) \le \min_{\alpha=\delta} \kappa_p(V_k) \le k^{1/p} n^{2/p'} \frac{(\sqrt{2}+1)3^{3/4}\psi_p(k,n)}{4\sqrt{k}\lfloor n/k\rfloor} \delta' S_{k,1}(\delta', 0); \tag{9.4}$$

and if $\delta \ge n^{1/[p'(k-1)]}$, then

$$\frac{\delta^{k-1} S_{k-1,p'}(\delta, 0)}{n^{1/p'}} \le \min_{\alpha=\delta} \kappa_p(V_k) \le k^{1/p} n^{1/p'} \frac{3^{3/4}\psi_p(k,n)\left(\cos\frac{\pi}{2n}\right)^{k-1}}{2\sqrt{k}\lfloor n/k\rfloor} (\delta')^k S_{k,1}(\delta', 0), \tag{9.5}$$

$$S_{k-1,p'}(n^{1/[p'(k-1)]}, 0) \le \min_{\alpha_j\le\delta} \kappa_p(V_k) \le \frac{(\sqrt{2}+1)3^{3/4}}{4} k^{1/p} n^{(2k-1)/[p'(k-1)]} \times \frac{\psi_p(k,n)}{\sqrt{k}\lfloor n/k\rfloor} S_{k,1}(n^{1/[p'(k-1)]}, 0). \tag{9.6}$$

Proof: 1) Observe that $\{V_k : \alpha = \delta\} \subset \{V_k : \alpha \le \delta\}$. So

$$\min_{\alpha\le\delta} \kappa_p(V_k) \le \min_{\alpha=\delta} \kappa_p(V_k). \tag{9.7}$$

This is the middle inequality in (9.4).

2) Apply (8.9) to the case $-a = b = \alpha \le \delta$ to obtain $\kappa_p(V_k) \ge S_{k-1,p'}(\alpha, 0) \ge S_{k-1,p'}(\delta, 0)$ because $S_{k-1,p'}(\alpha, 0)$ is decreasing in α by Theorem 2.1. This gives the first inequality in (9.4).

3) Apply (8.9) to the case $-a = b = \delta = \alpha$ to obtain the first inequality in (9.5).

4) We now prove the third inequality in (9.4) and the second inequality in (9.5). To this end, take $-a = b = \delta/\cos\frac{\pi}{2k} = \delta'$, and let V_k be constructed as in Lemma 9.1. Then

$$\alpha = \max|\alpha_j| = b\cos\frac{\pi}{2k} = \delta,$$
$$\delta \le \omega = b = \delta'.$$

Lemma 9.1 says that for this V_k

$$\begin{aligned} \frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\omega, 0)} &\le \omega \max\left\{1, \frac{1+\omega}{1+\omega^2}\right\} \frac{3^{3/4}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor} \\ &\le \begin{cases} \delta' \dfrac{(\sqrt{2}+1)3^{3/4}\psi_p(k,n)}{4\sqrt{k}\lfloor n/k\rfloor}, & \text{if } \delta \le n^{1/[p'(k-1)]}, \\ \delta' \dfrac{3^{3/4}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor}, & \text{if } \delta \ge n^{1/[p'(k-1)]}. \end{cases} \end{aligned}$$

Now employ $\|V_k\|_\infty \le k^{1/p} n^{2/p'}$ if $\delta \le n^{1/[p'(k-1)]}$ and $\|V_k\|_\infty \le k^{1/p} n^{1/p'} \delta^{k-1}$ if $\delta \ge n^{1/[p'(k-1)]}$ to complete the proof.

5) A proof of (9.6) can be done in the same way as for Theorem 9.1. ∎

Comments similar to those in Section 5 apply here.

10 Rectangular Vandermonde matrices whose nodes are in $[a, b]$ with $0 \le a < b$

Assume throughout this section

$$0 \le a < b.$$

For the same reason as in Section 7, assignments (7.1):

$$0 \le a = \min_j \alpha_j < b = \max_j \alpha_j = \alpha$$

will be kept throughout this section, too, unless otherwise explicitly stated.

Theorem 10.1 *Suppose $\alpha_j \ge 0$, and a and b defined by* (7.1)*. Then*

$$\begin{aligned}\kappa_p(V_k) \quad &\ge \quad \max\left\{S_{k-1,p'}(\omega,\tau), \frac{\alpha^{k-1}S_{k-1,p'}(\omega,\tau)}{n^{1/p'}}\right\}, &(10.1)\\ &\ge \quad \max\left\{S_{k-1,p'}(\alpha/2,1), \frac{\alpha^{k-1}S_{k-1,p'}(\alpha/2,1)}{n^{1/p'}}\right\}, &(10.2)\\ &= \quad \begin{cases} S_{k-1,p'}(\alpha/2,1), & \text{if } |\alpha| \le n^{1/[p'(k-1)]},\\ \dfrac{\alpha^{k-1}S_{k-1,p'}(\alpha/2,1)}{n^{1/p'}}, & \text{if } |\alpha| > n^{1/[p'(k-1)]}.\end{cases} &(10.3)\end{aligned}$$

Proof: (10.1) is (8.8) with (7.1). (10.2) holds because of $\omega \le \alpha/2$ and $\tau \le -1$, and the monotonicity of $S_{k-1,p'}(\omega,\tau)$ in its two arguments by Theorem 2.1. Also by Theorem 2.1, the first quantity within $\max\{\cdots\}$ in (10.2) is decreasing; while the second one is increasing. Therefore the right-hand side of (10.2) achieves its minimum when the two equal, i.e.,

$$n^{1/p'} = \alpha^{k-1} \quad \Rightarrow \quad \alpha = n^{1/[p'(k-1)]}$$

which yields (10.3). ∎

Lemma 10.1 *Let $W \in \mathbb{R}^{k\times k}$ be the Vandermonde matrix with nodes being the zeros of $T_k(x;\omega,\tau)$ defined with $0 \le a < b$, and define $V_k = Z$ as in* Lemma 8.3*. Then*

$$\frac{[\phi_p(k,n)]^{-1}}{k^{3/2}\lceil n/k\rceil}\,\frac{\frac{b-a}{2}\cos\frac{\pi}{2k}}{\left(1+\frac{a+b}{2}\right)} \le \frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\omega,\tau)} \le \frac{\psi_p(k,n)\,(b-a)}{2\sqrt{k}\lfloor n/k\rfloor\sqrt{(1+b)(1+a)}}. \tag{10.4}$$

Proof: Apply Theorem 4.5 to W here which is a $k \times k$ Vandermonde matrices with nodes being zeros of $T_k(x;\omega,\tau)$ defined with $0 \le a < b$, to get

$$\frac{\frac{b-a}{2}\cos\frac{\pi}{2k}}{k\left(1+\frac{a+b}{2}\right)} \le \frac{\|W^{-1}\|_\infty}{S_{k,1}(\omega,\tau)} \le \frac{b-a}{2k\sqrt{(1+b)(1+a)}}. \tag{10.5}$$

Now apply (8.16), noticing

$$\frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\omega,\tau)} = \frac{\mathsf{lub}_p(V_k)^{-1}}{\|W^{-1}\|_\infty}\cdot\frac{\|W^{-1}\|_\infty}{S_{k,1}(\omega,\tau)},$$

to complete the proof. ∎

Theorem 10.2

$$S_{k-1,p'}\left(n^{1/[p'(k-1)]}/2,1\right) \le \min_{\alpha_j\ge 0}\kappa_p(V_k) \le \frac{k^{1/p}n^{(2k-1)/[p'(k-1)]}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor\sqrt{1+n^{1/[p'(k-1)]}}} \times S_{k,1}(n^{1/[p'(k-1)]}/2,1). \tag{10.6}$$

Proof: The first inequality is true because the right-hand side of (10.2) achieves its minimum at $\alpha = n^{1/[p'(k-1)]}$. We now prove the second one. To this end, let V_k be constructed as in Lemma 10.1 with $0 = a < b = n^{1/[p'(k-1)]}$. Then

$$\mathsf{lub}_p(V_k)^{-1} \le \frac{\psi_p(k,n)\, n^{1/[p'(k-1)]}}{2\sqrt{k}\lfloor n/k\rfloor\sqrt{1+n^{1/[p'(k-1)]}}}\, S_{k,1}(n^{1/[p'(k-1)]}/2,1).$$

The proof will be completed by noticing that for this V_k, $\|V_k\|_p \le k^{1/p}n^{2/p'}$ by (8.5). ∎

Theorem 10.3 *Let $0 \le \delta$, and let $\delta' = 2\delta/(1+c) \le \delta$, where $c = \cos\frac{\pi}{2k}$. If $\delta < n^{1/(k-1)}$, then*

$$S_{k-1,p'}(\delta/2,1) \le \min_{0\le\alpha_j\le\delta}\kappa_p(V_k) \le \min_{\substack{0\le\alpha_j\le\delta\\ \alpha=\delta}}\kappa_p(V_k) \le \frac{k^{1/p}n^{2/p'}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor}\,\frac{\delta'}{\sqrt{1+\delta'}}\,S_{k,1}(\delta'/2,1); \tag{10.7}$$

and if $\delta \ge n^{1/(k-1)}$, then

$$\frac{\delta^{k-1}S_{k-1,p'}(\delta/2,1)}{n^{1/p'}} \le \min_{\substack{0\le\alpha_j\le\delta\\ \alpha=\delta}}\kappa_p(V_k) \le \frac{k^{1/p}n^{1/p'}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor}\left(\frac{1+c}{2}\right)^{k-1}\frac{(\delta')^k}{\sqrt{1+\delta'}}\,S_{k,1}(\delta'/2,1), \tag{10.8}$$

$$S_{k-1,p'}\left(n^{1/[p'(k-1)]}/2,1\right) \le \min_{0\le\alpha_j\le\delta}\kappa_p(V_k) \le \frac{k^{1/p}n^{(2k-1)/[p'(k-1)]}\psi_p(k,n)}{2\sqrt{k}\lfloor n/k\rfloor\sqrt{1+n^{1/[p'(k-1)]}}} \times S_{k,1}(n^{1/[p'(k-1)]}/2,1). \tag{10.9}$$

Proof: 1) Observe that $\{V_k : 0 \le \alpha_j \le \alpha = \delta\} \subset \{V_k : 0 \le \alpha_j \le \alpha \le \delta\}$. So

$$\min_{0\le\alpha_j\le\delta}\kappa_p(V_k) \le \min_{\substack{0\le\alpha_j\le\delta\\ \alpha=\delta}}\kappa_p(V_k). \tag{10.10}$$

This gives the middle inequality in (10.7).

2) Apply (10.3) to get

$$\kappa_p(V_k) \ge S_{k-1,p'}(\alpha/2,1) \ge S_{k-1,p'}(\delta/2,1)$$

because $S_{k-1,p'}(\alpha/2,1)$ is decreasing in α by Theorem 2.1. This gives the first inequality in (10.7).

3) Apply (10.3) to get the first inequality in (10.8).

4) We now prove the third inequality in (10.7) and the second inequality in (10.8). To this end, take $0 = a < b$ (to be determined by requiring $\alpha = \delta$), and let V_k be constructed as in Lemma 10.1. Then

$$\alpha = \max |\alpha_j| = b\frac{1+c}{2} = \delta, \quad \omega = b/2 = \delta'/2.$$

Lemma 10.1 says that for this V_k

$$\frac{\mathsf{lub}_p(V_k)^{-1}}{S_{k,1}(\omega, 0)} \le \frac{\psi_p(k,n)\, b}{2\sqrt{k}\lfloor n/k \rfloor \sqrt{1+b}} = \frac{\psi_p(k,n)\, \delta'}{2\sqrt{k}\lfloor n/k \rfloor \sqrt{1+\delta'}}$$

Now employ $\|V_k\|_p \le k^{1/p} n^{2/p'}$ if $\delta \le n^{1/[p'(k-1)]}$ and $\|V_k\|_p \le k^{1/p} n^{1/p'} \delta^{k-1}$ if $\delta \ge n^{1/[p'(k-1)]}$ to complete the proof.

5) A proof of (10.9) can be done in the same way as for Theorem 10.2. ∎

Comments similar to those in Section 7 apply here.

11 Confluent Vandermonde matrices

The standard Vandermonde matrix (1.1) is singular whenever $\alpha_i = \alpha_j$ for some $i \ne j$. It can be generalized to handle nodes not all of which are distinct. This leads to so-called *Confluent Vandermonde Matrices*, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by "differentiating" the previous column. Confluent Vandermonde matrices arises in Hermite interpolation [2]. Adopting the formulation in [14], we define *confluent Vandermonde matrix* V_c as follows.

Nodes $\{\alpha_j\}_{j=1}^n$ are ordered so that equal nodes are contiguous, i.e.,

$$\alpha_i = \alpha_j \quad (i < j) \quad \Rightarrow \quad \alpha_i = \alpha_{i+1} = \cdots = \alpha_j. \tag{11.1}$$

Define

$$V_c = (f_1(\alpha_1)\ f_2(\alpha_2)\ \cdots\ f_n(\alpha_n)), \tag{11.2}$$

where vector function $f_j(t)$ is defined recursively by

$$f_j(t) = \begin{cases} (1\ t\ \cdots\ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\ \frac{d}{dx} f_{j-1}(t), & \text{otherwise.} \end{cases} \tag{11.3}$$

As far as defining V_c is concerned, α_j can be real or complex. But as so far in this paper, we shall focus on real α_j. In what follows, we shall mainly emphasize on explaining a technique to derive lower bounds on $\kappa_p(V_c)$, rather than writing out every possible bounds we can get. For the sake of presentation, we assume, in addition to (11.1),

There are ℓ distinct nodes α_j, having multiplicities $k_1, k_2, \ldots, k_\ell$, respectively, where $k_1 + k_2 + \cdots + k_\ell = n$. This implies that the first k_1 α_j's are equal, the next k_2 α_j's are equal, and so on. Define $k_{\max} = \max_j k_j$.	(11.4)

All previously introduced notation still applies.

Lemma 11.1 *Assume* (11.1) *and* (11.4). *Then*

$$\|V_c\|_p \geq \max\left\{\ell^{1/p'}, \alpha^{n-1}\right\}. \tag{11.5}$$

Proof: Use $\|V_c\|_p \geq \|V_c^T e_1\|_{p'}$ and $\|V_c\|_p \geq \|V_c^T e_n\|_{p'}$ to get (11.5). ∎

Lemma 11.2 *For* $0 \leq k \leq n$,

$$\left|\frac{d}{dx^k} T_n(x;\omega,\tau)\right| \leq \frac{[n(n-1)\cdots(n-k+1)]^2}{\omega^k} \quad \textit{for } x \in [a,b]. \tag{11.6}$$

Proof: It follows from $T_n(x;\omega,\tau) = T_n(x/\omega+\tau) = T_n(t)$ that

$$\frac{d}{dx^k} T_n(x;\omega,\tau) = \frac{1}{\omega^k} T_n^{(k)}(t),$$

where $t \equiv t(x) = x/\omega + \tau$. It suffices to show that $|T_n^{(k)}(t)| \leq [n(n-1)\cdots(n-k+1)]^2$ for $t \in [-1,1]$ since $t(x)$ maps $x \in [a,b]$ to $t \in [-1,1]$. By Markov's Inequality [4, Page 233],

$$\begin{aligned} \max_{t\in[-1,1]} |T_n^{(k)}(t)| &\leq (n-k+1)^2 \max_{t\in[-1,1]} |T_n^{(k-1)}(t)| \\ &\leq \cdots \\ &\leq [n(n-1)\cdots(n-k+1)]^2 \max_{t\in[-1,1]} |T_n(t)| \\ &= [n(n-1)\cdots(n-k+1)]^2, \end{aligned}$$

as expected. ∎

Lemma 11.3 *Assume* (11.1) *and* (11.4). *Then*

$$\|V_c^{-1}\|_p \geq \min\left\{1, \left[\frac{\omega}{(n-1)^2}\right]^{k_{\max}-1}\right\} \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}. \tag{11.7}$$

Proof: Let v be the vector of the coefficients of the translated Chebyshev polynomial $T_{n-1}(x;\omega,\tau)$, i.e., $v = (a_{0\,n-1}\ a_{1\,n-1}\ \cdots\ a_{n-1\,n-1})^T$. Then

$$V_{\rm c}^T v = (T_{n-1}(\alpha_1;\omega,\tau)\ T'_{n-1}(\alpha_1;\omega)\ \cdots\ T_{n-1}^{(k_1-1)}(\alpha_1;\omega)\ \cdots\ \cdots)^T$$

which yields, by Lemma 11.2, for $1 \le p' < \infty$

$$\begin{aligned}
\|V_{\rm c}^T v\|_{p'}^{p'} &\le \sum_{j=1}^{\ell}\left(1^{p'} + \left[\frac{(n-1)^2}{\omega}\right]^{p'} + \cdots + \left[\frac{[(n-1)(n-2)\cdots(n-k_j+1)]^2}{\omega^{k_j-1}}\right]^{p'}\right) \\
&\le \sum_{j=1}^{\ell}\left(1 + \left[\frac{(n-1)^2}{\omega}\right]^{p'} + \left[\frac{[(n-1)^2]^2}{\omega^2}\right]^{p'} + \cdots + \left[\frac{[(n-1)^2]^{k_j-1}}{\omega^{k_j-1}}\right]^{p'}\right) \\
&\le \begin{cases} n, & \text{if } \omega \ge (n-1)^2, \\ n\left[\dfrac{(n-1)^2}{\omega}\right]^{p'(k_{\max}-1)}, & \text{otherwise} \end{cases} \\
&= n\max\left\{1, \left[\frac{(n-1)^2}{\omega}\right]^{p'(k_{\max}-1)}\right\}
\end{aligned} \tag{11.8, 11.9}$$

which gives

$$\|V_{\rm c}^T v\|_{p'} \le n^{1/p'}\max\left\{1, \left[\frac{(n-1)^2}{\omega}\right]^{k_{\max}-1}\right\}. \tag{11.10}$$

This is proved so far for $1 \le p' < \infty$, but it can be verified that (11.10) holds for $p' = \infty$, too. We therefore have

$$\begin{aligned}
\|V_{\rm c}^{-T}\|_{p'} &= \max_{u\in\mathbb{R}^n}\frac{\|u\|_{p'}}{\|V_{\rm c}^T u\|_{p'}} \ge \frac{\|v\|_{p'}}{\|V_{\rm c}^T v\|_{p'}} \\
&\ge \min\left\{1, \left[\frac{\omega}{(n-1)^2}\right]^{k_{\max}-1}\right\}\frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}},
\end{aligned}$$

as was to be shown. ∎

Theorem 11.1 *Assume* (11.1) *and* (11.4). *Then*

$$\kappa_p(V_{\rm c}) \ge \min\left\{1, \left[\frac{\omega}{(n-1)^2}\right]^{k_{\max}-1}\right\} \times \max\{\ell^{1/p'}, \alpha^{n-1}\}\cdot\frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}. \tag{11.11}$$

Proof: It is a consequence of Lemmas 11.1 and 11.3. ∎

As for how to apply Theorem 11.1 to any given $V_{\rm c}$, our comments right after Theorem 5.1 still work.

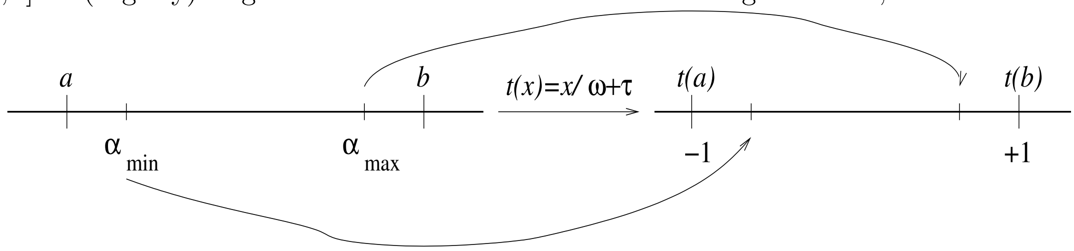
Remark **11.1** When $k_{\max} = 1$, i.e., $\ell = n$ and $k_1 = \cdots = k_n = 1$, $V_c = V$. Miraculously lower bound (11.11) for $\kappa_p(V_c)$ becomes the lower bound (5.9) for $\kappa_p(V)$. In general, we may also use (11.8), instead of (11.9), in estimating $\|V_c^{-1}\|_p$. The drawback is making the final lower bound on $\kappa_p(V_c)$ more complicated.

Remark **11.2** Applying Theorem 11.1 to cases $-a = b$ or $0 \le a < b$ will lead to explicitly computable lower bounds for $\kappa_p(V_c)$, as we did in Sections 5 and 7. Details are omitted. The next question is *are those bounds or some of them asymptotically best possible?* In Section 5 and 7, we answered similar questions for V with successful computations for V with translated Chebyshev nodes, thanks to the work by Gautschi [8]. But what now? It is conceivable that the key here would still be some computations for V_c with special nodes. This is yet another one of the many open problems we have put forward so far.

Remark **11.3** Lemma 11.3 is made possible by Lemma 11.2 which was proved with the help of Markov's Inequality. Another classical inequality for the same purpose is Bernstein's Inequality [4, Page 233], using which we can obtain the following: *For* $0 \le k \le n$, *if* $a < \alpha_{\min} < \alpha_{\max} < b$, *then*

$$\left|\frac{d}{dx^k} T_n(x;\omega,\tau)\right| \le \frac{n(n-1)\cdots(n-k+1)}{\left[\omega\sqrt{1-\left(\frac{\max\{b-\alpha_{\max},\,\alpha_{\min}-a\}}{\omega}\right)^2}\,\right]^k} \quad \textit{for } x \in [a,b], \tag{11.12}$$

where $\alpha_{\max} \equiv \alpha = \max_j \alpha_j$ *and* $\alpha_{\min} \equiv \min_j \alpha_j$. This inequality improves (11.6) in the numerator part, but has complications in the denominator, and also it requires the interval $[a,b]$ be (slightly) larger than the smallest interval containing all nodes, as follows.



This can be bad because larger $[a,b]$ will weaken the effectiveness of $S_{n,p'}(\omega,\tau)$ in the later bounds on $\kappa_p(V_c)$, for example $S_{n,p'}(\omega,\tau)$ is decreasing in ω. Therefore, as far as the asymptotical sharpness of derived lower bounds is concerned, using Markov's inequality is most likely a better choice.

Remark **11.4** Along similar lines to Sections 8 and 10, rectangular confluent Vandermonde matrices can be defined and lower bounds on their condition numbers can be established, too.

12 Possible extensions to Vandermonde-like matrix and complex Vandermonde matrices

This section outlines two possible extensions to (confluent) Vandermonde-like matrices and complex Vandermonde Matrices.

A Vandermonde-like matrix $\widetilde{V}$ is defined by

$$\widetilde{V} \stackrel{\text{def}}{=} \begin{pmatrix} p_0(\alpha_1) & p_0(\alpha_2) & \cdots & p_0(\alpha_n) \\ p_1(\alpha_1) & p_1(\alpha_2) & \cdots & p_1(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(\alpha_1^{n-1}) & p_{n-1}(\alpha_2^{n-1}) & \cdots & p_{n-1}(\alpha_n^{n-1}) \end{pmatrix}, \tag{12.1}$$

where p_i is a polynomial of degree i. For practical considerations, often these p_i form a family of orthogonal polynomials and thus satisfy a three-term recurrence relation, often α_i are the zeros of p_n in the family. As to the conditioning of V with orthogonal polynomials, Gautschi [11] derived an explicit formula for its condition number in Frobenius norm in terms of the associated Christoffel numbers. Reichel and Opfer [18] developed a progressive way to control the condition number for $\widetilde{V}$ by Chebyshev polynomials. However what we are going to outline is a technique to obtain a lower bound on $\kappa_\infty(\widetilde{V})$ in the most general setting, i.e., p_i is a polynomial of degree i, and all $\alpha_j \in [a, b]$.

Expand $T_n(x; \omega, \tau)$ as a linear combination of $p_0(x), p_1(x), \ldots, p_n(x)$:

$$T_n(x; \omega, \tau) = \widetilde{a}_{0n} p_0(x) + \widetilde{a}_{1n} p_1(x) + \cdots + \widetilde{a}_{nn} p_n(x),$$

and define, similarly to $S_{n,p}(\omega, \tau)$,

$$\widetilde{S}_{n,p}(\omega, \tau) = \left(\sum_{j=0}^{n} |\widetilde{a}_{jn}|^p \right)^{1/p}.$$

Then it can be shown that, using the lines of arguments from Lemma 5.1,

$$\|\widetilde{V}^{-1}\|_p \geq \frac{\widetilde{S}_{n-1,p'}(\omega, \tau)}{n^{1/p'}}. \tag{12.2}$$

Often getting a lower bound on $\|\widetilde{V}\|_p$ is rather straightforward, and with that a lower bound on $\kappa_p(\widetilde{V})$ is readily established. This technique obviously is adaptable to a confluent Vandermonde-like matrix whose definition can be found in [14].

Now we consider a complex Vandermonde matrix. We still use the symbol V but with complex α_j. Suppose that all α_j lie in the ellipse $\mathcal{E}$ with foci $a, b \in \mathbb{R}$, and semimajor axis R and semiminor axis r, as shown in Figure 12.1. It can be shown that [19, Page 203]

$$\max_{z \in \mathcal{E}} |T_n(z; \omega, \tau)| = T_n(z; \omega, \tau)|_{z=(b+a)/2+R} = T_n(R/\omega).$$

Therefore, resembling Lemma 5.1, we have

$$\|V^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}\, T_{n-1}(R/\omega)}. \tag{12.3}$$

What we have above is a brief description of possible extensions. More study along these lines will be pursued in future papers.

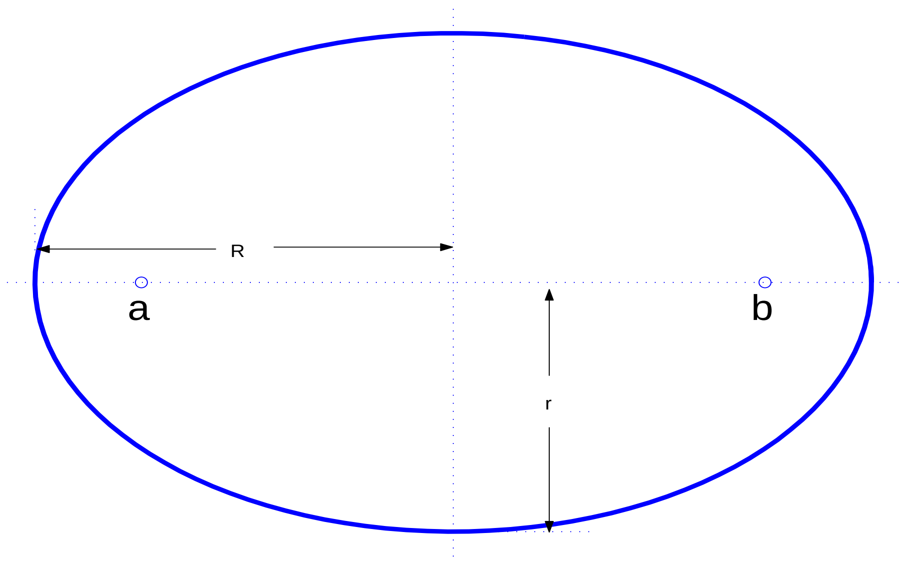


Figure 12.1: Ellipse $\mathcal{E}$ with real foci a and b, that contains all α_j

13 Concluding remarks, open problems

We have obtained a series of lower and upper bounds on the condition number $\kappa_p(V)$ of a real Vandermonde matrix V. These bounds are proved to be asymptotically optimal, except possibly the one in Theorem 5.1 in the case when interval $[a, b]$ is *not* one of the three kinds: 1) symmetrical ($-a = b$); 2) nonnegative ($a \geq 0$); 3) non-positive ($b \leq 0$). I.e., $a < 0 < b$ but $-a \neq b$. These bounds improve significantly those of previous ones by Gautschi and Ingese [12] and Tyrtyshnikov [20].

Our results led us to deduce the qualitative behaviors of optimally conditioned Vandermonde matrices as the largest absolute value α of all nodes varies, as shown in Figure 1.1 at the beginning of this paper. Our proofs yielded nearly optimally conditioned Vandermonde matrices in various circumstances.

We still do not know whether the lower bound in Theorem 5.1 is in general asymptotically optimal. Based on what we have here, what is left is the consideration of the interval $[a, b]$ with $a < 0 < b$ but $-a \neq b$. We suspect it would be, but no proof yet. The crucial step in answering this question would still lie somewhat in the estimation of the condition number of the Vandermonde matrix with the translated Chebyshev nodes.

We believe various $\mathcal{O}_n(\cdots)$ in this paper could be replaced by $\mathcal{O}(\cdots)$. See the arguments that lead to Conjecture 6.1.

Another problem for which we do not yet have a completely satisfactory solution is finding rather accurate estimations for, or better yet – an explicit formula for $S_{n,1}(\omega, \tau)$ for interval $[a, b]$ with $a < 0 < b$ but $-a \neq b$.

We do not know whether the lower bound in Theorem 11.1 for confluent Vandermonde matrix V_c and those implied by it for symmetrical or nonnegative interval $[a, b]$ are asymptotically optimal.

We demonstrated how the technique could be applied to Vandermonde-like matrices and to complex Vandermonde matrices. Many questions remain. Extensions to complex Vandermonde matrices with nodes sitting on a straight line passing through the origin in the complex plane are a bit obvious and perhaps even trivial.

Extensions are certainly possible for rectangular confluent Vandermonde matrices similarly defined as in Section 8.

Answering these problems will be the subject of future research.

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Lemma 3.1 is buggy, namely inequality (3.9) *for* $i < n$ *may be incorrect due to the gap in deriving* (3.9) *from* (3.8). *But* (3.9) *for* $i = n$ *is true, and as a result, condition number bounds in the the rest of this paper are true. Also the condition* (3.5)*:* $(b-a)/2 < 2$ *is unnecessary. The claim* $cond_\infty(V) \geq cond_2(V)$ *following* (4.4) *is also problematic.*

A Asymptotical behavior of $\int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta$

For $n = 1$ and $n = 2$, we have

$$\int_0^{\pi/2} [\cos\theta]^{1-1}\,d\theta = \frac{\pi}{2}, \quad \int_0^{\pi/2} [\cos\theta]^{2-1}\,d\theta = 1.$$

By integration-by-parts,

$$\begin{aligned}
\int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta &= \int_0^{\pi/2} [\cos\theta]^{n-2}\,d\sin\theta \\
&= [\cos\theta]^{n-2}\sin\theta\Big|_{\theta=0}^{\pi/2} + \int_0^{\pi/2} [\sin\theta]^2 (n-2)[\cos\theta]^{n-3}\,d\theta \\
&= \int_0^{\pi/2} (1-[\cos\theta]^2)(n-2)[\cos\theta]^{n-3}\,d\theta \\
&= (n-2)\int_0^{\pi/2} [\cos\theta]^{n-3}\,d\theta - (n-2)\int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta &= \frac{n-2}{n-1}\int_0^{\pi/2} [\cos\theta]^{n-3}\,d\theta \\
&= \cdots \\
&= \frac{(n-2)!!}{(n-1)!!}\cdot \begin{cases} \frac{\pi}{2}, & \text{for odd } n, \\ 1, & \text{for even } n, \end{cases}
\end{aligned} \tag{A.1}$$

where $N!!$ is the product of all every other positive integers, starting at N down to 1, e.g., $5!! = 5\cdot 3\cdot 1$, $6!! = 6\cdot 4\cdot 2$. Now for odd $n = 2m-1$,

$$\frac{(n-2)!!}{(n-1)!!} = \frac{(2m-3)!!}{(2m-2)!!} = \frac{(2m-2)!}{[(2m-2)!!]^2} = \frac{(2m-1)!}{2^{2m-2}[(m-1)!]^2}\cdot\frac{1}{2m-1}; \tag{A.2}$$

and for even $n = 2m$,

$$\frac{(n-2)!!}{(n-1)!!} = \frac{(2m-2)!!}{(2m-1)!!} = \frac{[(2m-2)!!]^2}{(2m-1)!} = \frac{2^{2m-2}[(m-1)!]^2}{(2m-1)!}. \tag{A.3}$$

By Stirling's asymptotic formula [1, Page 18], we have

$$\frac{(2m-1)!}{2^{2m-2}[(m-1)!]^2} = \frac{\Gamma(2m)}{2^{2m-2}[\Gamma(m)]^2} \sim \frac{\sqrt{2\pi}\,(2m)^{2m-1/2}\,e^{-2m}}{2^{2m-2}\left(\sqrt{2\pi}\,m^{m-1/2}\,e^{-m}\right)^2} = \frac{2^{3/2}}{\sqrt{2\pi}}\,m^{1/2}.$$

Therefore it follows from (A.2) and (A.3) that

$$\frac{(n-2)!!}{(n-1)!!} \sim \begin{cases} \dfrac{2^{3/2}}{\sqrt{2\pi}}\left(\dfrac{n}{2}\right)^{1/2}\dfrac{1}{n} = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\sqrt{n}}, & \text{for odd } n, \\ \dfrac{\sqrt{2\pi}}{2^{3/2}}\left(\dfrac{n}{2}\right)^{-1/2} = \sqrt{\dfrac{\pi}{2}}\dfrac{1}{\sqrt{n}}, & \text{for even } n. \end{cases}$$

This together with (A.1) gives

$$\int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta \sim \sqrt{\frac{\pi}{2n}}. \tag{A.4}$$

B $\Lambda_n(0)$ and $\Lambda_n(1)$

We have

$$\Lambda_n(0) = \sum_{j=1}^{n} |\cos\theta_j|^{n-1} = 2\sum_{j=1}^{\lfloor n/2\rfloor} [\cos\theta_j]^{n-1} = 2\frac{n}{\pi}\cdot\frac{\pi}{n}\sum_{j=1}^{\lfloor n/2\rfloor}[\cos\theta_j]^{n-1}.$$

Therefore

$$\begin{aligned} \frac{\pi}{2n}\Lambda_n(0) &\le \int_{-\pi/(2n)}^{\pi/2} [\cos\theta]^{n-1}\,d\theta \\ &\le \int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta + \frac{\pi}{2n}, \\ \frac{\pi}{2n}\Lambda_n(0) &\ge \int_{\pi/(2n)}^{\pi/2} [\cos\theta]^{n-1}\,d\theta \\ &\ge \int_0^{\pi/2} [\cos\theta]^{n-1}\,d\theta - \frac{\pi}{2n}. \end{aligned}$$

They yield

$$\frac{2n}{\pi}\int_0^{\pi/2}[\cos\theta]^{n-1}\,d\theta - 1 \le \Lambda_n(0) \le \frac{2n}{\pi}\int_0^{\pi/2}[\cos\theta]^{n-1}\,d\theta + 1. \tag{B.1}$$

Now with the help of (A.4), we get

$$\Lambda_n(0) \sim \frac{2n}{\pi}\cdot\sqrt{\frac{\pi}{2n}} = \sqrt{\frac{2n}{\pi}}. \tag{B.2}$$

For $\Lambda_n(1)$, we have

$$\Lambda_n(1) = \sum_{j=1}^{n} (\cos\theta_j + 1)^{n-1} = 2^{n-1} \sum_{j=1}^{n} \left(\cos\frac{\theta_j}{2}\right)^{2(n-1)} = 2^{n-1} \frac{n}{\pi} \cdot \frac{\pi}{n} \sum_{j=1}^{n} \left(\cos\frac{\theta_j}{2}\right)^{2(n-1)}.$$

Therefore

$$\begin{aligned}
\frac{\pi}{n\,2^{n-1}} \Lambda_n(1) &\leq \int_{-\pi/(2n)}^{\pi} \left(\cos\frac{\theta}{2}\right)^{2(n-1)} d\theta \\
&\leq \int_{0}^{\pi} \left(\cos\frac{\theta}{2}\right)^{2(n-1)} d\theta + \frac{\pi}{2n} \\
&\leq 2\int_{0}^{\pi/2} (\cos\theta)^{2(n-1)}\, d\theta + \frac{\pi}{2n}, \\
\frac{\pi}{n\,2^{n-1}} \Lambda_n(1) &\geq \int_{\pi/(2n)}^{\pi} \left(\cos\frac{\theta}{2}\right)^{2(n-1)} d\theta \\
&\geq \int_{0}^{\pi} \left(\cos\frac{\theta}{2}\right)^{2(n-1)} d\theta - \frac{\pi}{2n} \\
&\geq 2\int_{0}^{\pi/2} (\cos\theta)^{2(n-1)}\, d\theta - \frac{\pi}{2n}.
\end{aligned}$$

They yield

$$\frac{n\,2^n}{\pi} \int_{0}^{\pi/2} (\cos\theta)^{2(n-1)}\, d\theta - 2^{n-2} \leq \Lambda_n(1) \leq \frac{n\,2^n}{\pi} \int_{0}^{\pi/2} (\cos\theta)^{2(n-1)}\, d\theta + 2^{n-2}. \qquad \text{(B.3)}$$

Now with the help of (A.4), we get

$$\Lambda_n(1) \sim \frac{n\,2^n}{\pi} \cdot \sqrt{\frac{\pi}{4n}} = \sqrt{\frac{n}{\pi}}\, 2^{n-1}. \qquad \text{(B.4)}$$