

Vandermonde Matrices with Chebyshev Nodes¹

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ABSTRACT

For $n \times n$ Vandermonde matrix $V_n = (\alpha_j^{i-1})_{1 \leq i, j \leq n}$ with translated Chebyshev zero nodes, it is discovered that V_n^T admits an explicit QR decomposition with the R -factor consisting of the coefficients of the translated Chebyshev polynomials of degree less than n . This decomposition then leads to an exact expression for the condition number of its submatrix $V_{k,n} = (\alpha_j^{i-1})_{1 \leq i \leq k, 1 \leq j \leq n}$ (so-called rectangular Vandermonde matrix), bounds on individual singular value, and more. It is explained that how these results can be used to establish asymptotically optimal lower bounds on condition numbers of real rectangular Vandermonde matrices and nearly optimal conditioned real rectangular Vandermonde matrices on a given interval. Extensions are also made for V_n with nodes being zeros of any (translated) orthogonal polynomials other than Chebyshev ones.

It is also discovered that for V_{n+1} with translated Chebyshev extreme nodes, V_{n+1}^T admits an explicit QR-like decomposition as well. This QR-like decomposition also yields similar conclusions to those for V_n with translated Chebyshev zero nodes.

Applications to the study of sharpness in existing error bounds for the conjugate gradient method and the minimal residual method for linear systems and the symmetric Lanczos method for eigenvalue problems are also discussed.

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Abstract

For $n \times n$ Vandermonde matrix $V_n = (\alpha_j^{i-1})_{1 \leq i, j \leq n}$ with translated Chebyshev zero nodes, it is discovered that V_n^T admits an explicit QR decomposition with the R -factor consisting of the coefficients of the translated Chebyshev polynomials of degree less than n . This decomposition then leads to an exact expression for the condition number of its submatrix $V_{k,n} = (\alpha_j^{i-1})_{1 \leq i \leq k, 1 \leq j \leq n}$ (so-called rectangular Vandermonde matrix), bounds on individual singular value, and more. It is explained that how these results can be used to establish asymptotically optimal lower bounds on condition numbers of real rectangular Vandermonde matrices and nearly optimal conditioned real rectangular Vandermonde matrices on a given interval. Extensions are also made for V_n with nodes being zeros of any (translated) orthogonal polynomials other than Chebyshev ones.

It is also discovered that for V_{n+1} with translated Chebyshev extreme nodes, V_{n+1}^T admits an explicit QR-like decomposition as well. This QR-like decomposition also yields similar conclusions to those for V_n with translated Chebyshev zero nodes.

Applications to the study of sharpness in existing error bounds for the conjugate gradient method and the minimal residual method for linear systems and the symmetric Lanczos method for eigenvalue problems are also discussed.

1 Introduction

Given n numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ called *nodes*, the associated *Vandermonde Matrix* is defined as

$$V_n \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}. \quad (1.1)$$

In [11], we established various asymptotically optimal lower bounds on condition numbers of real V_n . The key idea was to use the coefficients of Chebyshev polynomials of the first kind to arrive at lower bounds on the norms of V_n^{-1} and the explicit computation of the ℓ_∞ -operator norm of V_n^{-1} with (translated) Chebyshev zero nodes with the help of Gautschi's formula [6]. Two similar bounds were also obtained by Beckermann [2].

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This paper is devoted to the study of V_n with (translated) Chebyshev zero and extreme nodes. Various inequalities involving its singular values are obtained, as well as results that have applications to the convergence rate of the Conjugate Gradient method for positive definite linear systems, the minimal residual method, and the symmetric Lanczos algorithm for eigenvalue problems.

By default, we denote and order the singular values of an n -by- m matrix X as

$$\sigma_1(X) \leq \sigma_2(X) \leq \cdots \leq \sigma_{\min\{m,n\}}(X). \quad (1.2)$$

Matrix condition numbers are usually defined for square matrices, but they can be extended without difficulty to non-square matrices. We define X 's Frobenius condition number by

$$\kappa_F(X) \stackrel{\text{def}}{=} \sqrt{\sum_{j=1}^{\min\{m,n\}} [\sigma_j(X)]^2} \sqrt{\sum_{j=1}^{\min\{m,n\}} \frac{1}{[\sigma_j(X)]^2}}.$$

The first factor in the right-hand side is just X 's Frobenius norm $\|X\|_F$. If the rank of X is less than $\min\{m,n\}$, then $\sigma_{\min\{m,n\}}(X) = 0$ and thus $\kappa_F(X) = \infty$. Later in Section 3, ℓ_p -condition number $\kappa_p(X)$ will be defined, too, for $1 \leq p \leq \infty$.

The rest of this paper is organized as follows. Section 2 briefly reviews relevant material related to Chebyshev polynomials of the first kind. In Section 3, a lower bound on the ℓ_p -condition number of V_n 's submatrix $V_{k,n} = (\alpha_j^{i-1})_{1 \leq i \leq k, 1 \leq j \leq n}$ (so-called *rectangular Vandermonde matrix*) with all $\alpha_j \in [\alpha, \beta]$ (a given interval) is established. This bound, combined with later results on $V_{k,n}$ with translated Chebyshev nodes, leads to asymptotically optimal lower bounds for the case either $-\alpha = \beta$ or $\alpha\beta \geq 0$. Section 4 is devoted to $V_{k,n}$ with translated Chebyshev zero nodes, while Section 5 to $V_{k,n}$ with translated Chebyshev extreme nodes. Possible extensions to $V_{k,n}$ with translated zeros of an orthogonal polynomial are outlined in Section 6. Finally we give some conclusions in Section 7.

Notation. Throughout this paper, $\mathbb{C}^{n \times m}$ is the set of all $n \times m$ complex matrices, $\mathbb{C}^n = \mathbb{C}^{n \times 1}$, and $\mathbb{C} = \mathbb{C}^1$. Similarly define $\mathbb{R}^{n \times m}$, \mathbb{R}^n , and \mathbb{R} except replacing the word *complex* by *real*. I_n (or simply I if its dimension is clear from the context) is the $n \times n$ identity matrix, and e_j is its j th column. The superscript “.” takes conjugate transpose while “ T ” takes transpose only. We shall also adopt MATLAB-like convention to access the entries of vectors and matrices. $i : j$ is the set of integers from i to j inclusive and $i : i = \{i\}$. For vector u and matrix X , $u_{(j)}$ is u 's j th entry, $X_{(i,j)}$ is X 's (i,j) th entry, $\text{diag}(u)$ is the diagonal matrix with $(\text{diag}(u))_{(j,j)} = u_{(j)}$; X 's submatrices $X_{(k:\ell,i:j)}$, $X_{(k:\ell,:)}$, and $X_{(:,i:j)}$ consist of intersections of row k to row ℓ and column i to column j , row k to row ℓ , and column i to column j , respectively. $\lceil \xi \rceil$ is the largest integer that is no bigger than ξ ; while $\lfloor \xi \rfloor$ is the smallest integer that is no less than ξ .

Some of the estimates for condition numbers in this paper are not intended to be best possible but rather to correctly show their asymptotical speeds as k and n goes to ∞ . For this purpose, we shall use

$$a_n \stackrel{n}{\sim} b_n \text{ to mean that there are constants } c_1, c_2, d_1, \text{ and } d_2 \text{ such that } c_1 n^{d_1} \leq a_n/b_n \leq c_2 n^{d_2}, \text{ and } a_n \sim b_n \text{ if } a_n/b_n \rightarrow 1 \text{ as } n \rightarrow \infty.$$

2 Chebyshev polynomials

The n th Chebyshev polynomial of the 1st kind is

$$T_n(t) = \cos(n \arccos t) \quad \text{for } |t| \leq 1, \quad (2.1)$$

$$= \frac{1}{2} \left(t + \sqrt{t^2 - 1} \right)^n + \frac{1}{2} \left(t - \sqrt{t^2 - 1} \right)^n \quad \text{for } |t| \geq 1. \quad (2.2)$$

It frequently shows up in numerical analysis and computations because of its numerous nice properties, for example $|T_n(t)| \leq 1$ for $|t| \leq 1$ and $|T_n(t)|$ grows extremely fast for $|t| > 1$. It is known (see, e.g., [12])

$$\left| T_n \left(\frac{1+t}{1-t} \right) \right| \equiv \left| T_n \left(\frac{t+1}{t-1} \right) \right| = \frac{1}{2} [\Delta_t^n + \Delta_t^{-n}] \quad \text{for } 1 \neq t > 0, \quad (2.3)$$

where

$$\Delta_t \stackrel{\text{def}}{=} \frac{\sqrt{t} + 1}{|\sqrt{t} - 1|} \quad \text{for } 1 \neq t > 0. \quad (2.4)$$

Given two (real or complex) numbers $\omega \neq 0$ and τ , the n th *Translated Chebyshev Polynomial* in x of degree n is defined by

$$T_n(x; \omega, \tau) \stackrel{\text{def}}{=} T_n(x/\omega + \tau), \quad (2.5)$$

$$= a_{nn}x^n + a_{n-1n}x^{n-1} + \cdots + a_{1n}x + a_{0n}, \quad (2.6)$$

where $a_{jn} \equiv a_{jn}(\omega, \tau)$ are functions of ω and τ in (2.18). Their explicit dependence on ω and τ is often suppressed for convenience. Define

$$\text{Chebyshev zero nodes: } t_{jn} = \cos \theta_{jn}, \theta_{jn} = \frac{2j-1}{2n}\pi, 1 \leq j \leq n, \quad (2.7)$$

$$\text{translated Chebyshev zero nodes: } t_{jn}^{\text{tr}} = \omega(t_{jn} - \tau), 1 \leq j \leq n. \quad (2.8)$$

It can be seen that t_{jn} ($1 \leq j \leq n$) are the zeros of $T_n(t)$, while t_{jn}^{tr} ($1 \leq j \leq n$) are the zeros of $T_n(x; \omega, \tau)$. Define

$$\text{Chebyshev extreme nodes: } \tau_{jn} = \cos \vartheta_{jn}, \vartheta_{jn} = \frac{j}{n}\pi, 0 \leq j \leq n, \quad (2.9)$$

$$\text{translated Chebyshev extreme nodes: } \tau_{jn}^{\text{tr}} = \omega(\tau_{jn} - \tau), 0 \leq j \leq n. \quad (2.10)$$

τ_{jn} ($0 \leq j \leq n$) are the extreme points of $T_n(t)$ in $[-1, 1]$. For integer $m \geq 1$, define upper triangular $R_m \in \mathbb{C}^{m \times m}$, a matrix-valued function in ω and τ , as

$$R_m \equiv R_m(\omega, \tau) \stackrel{\text{def}}{=} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0m-1} \\ & a_{11} & a_{12} & \cdots & a_{1m-1} \\ & & a_{22} & \cdots & a_{2m-1} \\ & & & \ddots & \vdots \\ & & & & a_{m-1m-1} \end{pmatrix}, \quad (2.11)$$

i.e., the j th column consists of the coefficients of $T_{j-1}(x; \omega, \tau)$. In [11], $S_{n,p}(\omega, \tau)$ is defined by

$$S_{n,p}(\omega, \tau) = \left(\sum_{j=0}^n |a_{jn}|^p \right)^{1/p} \quad \text{for } 1 \leq p \leq \infty.$$

Also explicit formulas were found¹ for $p = 1$ and $\tau = 0$:

$$S_{n,1}(\omega, 0) = \frac{1}{2} \left[\left(\frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n + \left(\frac{1}{|\omega|} - \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n \right], \quad (2.12)$$

$$\sim \frac{1}{2} \left(\frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n, \quad (2.13)$$

and for all real τ with $|\tau| \geq 1$:

$$S_{n,1}(\omega, \tau) = \frac{1}{2} \left[\left(\frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left(\frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^n + \frac{1}{2} \left[\left(\frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left(\frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^{-n}, \quad (2.14)$$

$$\sim \frac{1}{2} \left[\left(\frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left(\frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^n. \quad (2.15)$$

No explicit formula or tight bounds are known for other τ , however. For $p \neq 1$, $S_{n,p}(\omega, \tau)$ relates to $S_{n,1}(\omega, \tau)$ by inequalities

$$(n+1)^{-1/p'} S_{n,1}(\omega, \tau) \leq S_{n,p}(\omega, \tau) \leq S_{n,1}(\omega, \tau), \quad (2.16)$$

$$\lceil (n+1)/2 \rceil^{-1/p'} S_{n,1}(\omega, 0) \leq S_{n,p}(\omega, 0) \leq S_{n,1}(\omega, 0), \quad (2.17)$$

where $1/p + 1/p' = 1$.

In the rest of this paper, by default, $\omega \neq 0$ and τ are two prescribed (real or complex) numbers, but when there is an interval $[\alpha, \beta]$ in the context, they are given by

$$\omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = -\frac{\alpha + \beta}{\beta - \alpha}. \quad (2.18)$$

The linear transformation

$$t(x) = \frac{x}{\omega} + \tau = \frac{2}{\beta - \alpha} \left(x - \frac{\alpha + \beta}{2} \right) \quad (2.19)$$

maps $x \in [\alpha, \beta]$ one-to-one and onto $t \in [-1, 1]$. The inverse transformation is $x(t) = \omega(t - \tau)$.

3 A general lower bound for $V_{k,n}$

This section concerns V_n with all $\alpha_j \in [\alpha, \beta]$ but otherwise general. The results will be used in the later sections.

¹Although in [11] these formulas were established with an interval $[\alpha, \beta]$ in the context (see (2.18)), the proofs there can be easily modified for ω and τ that have nothing to do with some interval $[\alpha, \beta]$.

We start by defining ℓ_p vector and operator norm. Given $1 \leq p \leq \infty$, the ℓ_p -norm of vector u and the ℓ_p -operator norm of matrix X are defined as

$$\|u\|_p = \left(\sum_{j=1}^n |u_{(j)}|^p \right)^{1/p}, \quad \|X\|_p = \max_{u \neq 0} \frac{\|Xu\|_p}{\|u\|_p}.$$

It is proved that [10] $\|X\|_p = \|X^T\|_{p'}$, where $1/p + 1/p' = 1$.

Let, throughout this paper,

$$V_{k,n} = (V_n)_{(:,1:k)}$$

be the submatrix of the first k rows of V_n . We define²

$$\text{lub}_p(V_{k,n}) \stackrel{\text{def}}{=} \min_{u \neq 0} \frac{\|V_{k,n}^T u\|_{p'}}{\|u\|_{p'}}, \quad \kappa_p(V_{k,n}) \stackrel{\text{def}}{=} \frac{\|V_{k,n}\|_p}{\text{lub}_p(V_{k,n})}. \quad (3.1)$$

Such definition is unlikely new, and is consistent with the case for $p = 2$ and the square matrix case. In fact $\text{lub}_2(V_{k,n}) = \sigma_1(V_{k,n})$, $V_{k,n}$'s smallest singular value, and for $k = n$, it can be shown that $\text{lub}_p(V_n) = \|V_n^{-1}\|_{p'}^{-1}$. We claim

$$\text{lub}_p(V_{k,n}) \leq \frac{n^{1/p'}}{S_{k-1,p'}(\omega, \tau)}. \quad (3.2)$$

Let v be the vector of the coefficients of $T_{k-1}(x; \omega, \tau) \equiv T_{k-1}(x/\omega + \tau)$ such that $v_{(j+1)} = a_{j,k-1}$ for $0 \leq j \leq k-1$. Then

$$V_{k,n}^T v = (T_{k-1}(\alpha_1/\omega + \tau) \ T_{k-1}(\alpha_2/\omega + \tau) \ \cdots \ T_{k-1}(\alpha_n/\omega + \tau))^T$$

which yields $\|V_{k,n}^T v\|_{p'} \leq n^{1/p'}$ because $|T_{k-1}(x/\omega + \tau)| \leq 1$ for $x \in [\alpha, \beta]$. We therefore have

$$\text{lub}_p(V_{k,n}) = \min_{u \in \mathbb{R}^k} \frac{\|V_{k,n}^T u\|_{p'}}{\|u\|_{p'}} \leq \frac{\|V_{k,n}^T v\|_{p'}}{\|v\|_{p'}} \leq \frac{n^{1/p'}}{\left(\sum_{j=0}^{k-1} |a_{j,k-1}|^{p'} \right)^{1/p'}},$$

as expected. We have proved the following theorem.

Theorem 3.1 For V_n with all nodes $\alpha_j \in [\alpha, \beta]$,

$$\kappa_p(V_{k,n}) \geq \|V_{k,n}\|_p \frac{S_{k-1,p'}(\omega, \tau)}{n^{1/p'}}, \quad (3.3)$$

where ω and τ are defined as in (2.18).

Specializing this theorem to the case $-\alpha = \beta$ or $\alpha\beta \geq 0$, we have

Theorem 3.2 Let V_n be with all nodes $\alpha_j \in [\alpha, \beta]$, and suppose $\max_j |\alpha_j| \geq \eta \max\{|\alpha|, |\beta|\}$ for some $\eta > 0$.

²For matrix $X \in \mathbb{C}^{n \times k}$ and $k < n$, it should be defined as $\text{lub}_p(X) = \min_{u \neq 0} \frac{\|Xu\|_p}{\|u\|_p}$.

1. If $-\alpha = \beta$, then

$$\kappa_p(V_{k,n}) \geq \max\{1, \eta^{k-1}\beta^{k-1}\} \frac{S_{k-1,1}(\beta, 0)}{[k/2]^{1/p} n^{1/p'}} \quad (3.4)$$

$$\begin{aligned} &\sim \frac{1}{2[k/2]^{1/p} n^{1/p'}} \times \\ &\max \left\{ \left(\frac{1}{\beta} + \sqrt{1 + \frac{1}{\beta^2}} \right)^{k-1}, \eta^{k-1} \left(\beta + \sqrt{1 + \beta^2} \right)^{k-1} \right\}. \end{aligned} \quad (3.5)$$

2. If $0 \leq \alpha < \beta$, then

$$\kappa_p(V_{k,n}) \geq \max\{1, \eta^{k-1}\beta^{k-1}\} \frac{S_{k-1,1}(\omega, \tau)}{k^{1/p} n^{1/p'}} \quad (3.6)$$

$$\geq \max\{1, \eta^{k-1}\beta^{k-1}\} \frac{S_{k-1,1}(\beta/2, 1)}{k^{1/p} n^{1/p'}} \quad (3.7)$$

$$\begin{aligned} &\sim \frac{1}{2k^{1/p} n^{1/p'}} \times \max \left\{ \left(\frac{2}{\beta} + 1 + 2\sqrt{\frac{1}{\beta^2} + \frac{1}{\beta}} \right)^{k-1}, \right. \\ &\left. \eta^{k-1} \left(2 + \beta + 2\sqrt{1 + \beta} \right)^{k-1} \right\}. \end{aligned} \quad (3.8)$$

Proof: Since $\max_j |\alpha_j| \geq \eta \max\{|\alpha|, |\beta|\}$, we have

$$\begin{aligned} \|V_{k,n}\|_p &\geq \max_j \|V_{k,n}^T e_j\|_{p'} = \max_j \left(\sum_{i=0}^{k-1} |\alpha_j|^{ip'} \right)^{1/p'} \\ &\geq \max\{1, \max_j |\alpha_j|^{k-1}\} \\ &\geq \max\{1, \eta^{k-1} |\alpha|^{k-1}, \eta^{k-1} |\beta|^{k-1}\}. \end{aligned} \quad (3.9)$$

Now if $-\alpha = \beta$, then $\omega = \beta$ and $\tau = 0$, and by (2.13) and (2.17)

$$\begin{aligned} S_{k-1,p'}(\beta, 0) &\geq [k/2]^{-1/p} S_{k-1,1}(\beta, 0) \\ &\sim [k/2]^{-1/p} \frac{1}{2} \left(\frac{1}{\beta} + \sqrt{1 + \frac{1}{\beta^2}} \right)^{k-1}. \end{aligned}$$

This, together with Theorem 3.1 and (3.9), lead to (3.4) and (3.5).

If $0 \leq \alpha < \beta$, then $\omega = (\beta - \alpha)/2 \leq \beta/2$ and $\tau \leq -1$. Since $S_{k-1,p'}(\omega, \tau) = S_{k-1,p'}(|\omega|, |\tau|)$ and it is increasing in $|\tau|$ and decreasing in $|\omega|$ [11], we have by (2.15) and (2.16)

$$\begin{aligned} S_{k-1,p'}(\omega, \tau) &\geq k^{-1/p} S_{k-1,1}(\omega, \tau) \\ &\geq k^{-1/p} S_{k-1,1}(\beta/2, 1) \\ &\sim k^{-1/p} \frac{1}{2} \left(\frac{2}{\beta} + 1 + 2\sqrt{\frac{1}{\beta^2} + \frac{1}{\beta}} \right)^{k-1}. \end{aligned}$$

This, together with Theorem 3.1 and (3.9), lead to (3.6) – (3.8). ■

4 V_n with Chebyshev zero nodes

In this section, V_n has the translated Chebyshev zero nodes $\alpha_j = t_{jn}^{\text{tr}}$ ($1 \leq j \leq n$), except possibly those $V_{k,n}$ in Theorem 4.3. It can be proved that [12]

$$V_n^T R_n = \mathbf{T}_n \stackrel{\text{def}}{=} \begin{pmatrix} T_0(t_{1n}) & T_1(t_{1n}) & T_2(t_{1n}) & \cdots & T_{n-1}(t_{1n}) \\ T_0(t_{2n}) & T_1(t_{2n}) & T_2(t_{2n}) & \cdots & T_{n-1}(t_{2n}) \\ \vdots & \vdots & \vdots & & \vdots \\ T_0(t_{nn}) & T_1(t_{nn}) & T_2(t_{nn}) & \cdots & T_{n-1}(t_{nn}) \end{pmatrix}, \quad (4.1)$$

$$\mathbf{T}_n^T \mathbf{T}_n = (n/2) \text{diag}(2, 1, 1, \dots, 1). \quad (4.2)$$

Notice that \mathbf{T}_n is real while V_n and R_n may be complex if ω or τ is. Equations (4.1) and (4.2) essentially give a QR decomposition for V_n^T after normalizing \mathbf{T}_n 's columns to have unit norm. Extracting the first k columns from the both sides of $V_n^T = \mathbf{T}_n R_n^{-1}$ yields the following theorem [12].

Theorem 4.1 *Let V_n have the translated Chebyshev zero nodes $\alpha_j = t_{jn}^{\text{tr}}$ ($1 \leq j \leq n$), and let upper triangular R_k be defined as in (2.11) and \mathbf{T}_n as in (4.1). Then $V_{k,n}^T = (\mathbf{T}_n)_{(:,1:k)} R_k^{-1}$.*

4.1 Condition number $\kappa_{\text{F}}(V_{k,n})$

By Theorem 4.1, we have³

$$\begin{aligned} \bar{V}_{k,n} V_{k,n}^T &= R_k^{-*} [(\mathbf{T}_n)_{(:,1:k)}]^T (\mathbf{T}_n)_{(:,1:k)} R_k^{-1} \\ &= R_k^{-*} (\mathbf{T}_n^T \mathbf{T}_n)_{(1:k,1:k)} R_k^{-1} \\ &= (n/2) R_k^{-*} \text{diag}(2, 1, 1, \dots, 1) R_k^{-1}, \end{aligned} \quad (4.3)$$

$$(\bar{V}_{k,n} V_{k,n}^T)^{-1} = (2/n) R_k \text{diag}(2^{-1}, 1, 1, \dots, 1) R_k^*, \quad (4.4)$$

where $\bar{V}_{k,n}$ is its complex conjugate. Consequently,

$$\begin{aligned} \sum_j [\sigma_j(V_{k,n})]^2 &= \frac{n}{2} \text{trace} (R_k^{-*} \text{diag}(2, 1, 1, \dots, 1) R_k^{-1}) \\ &= \frac{n}{2} \left\| \text{diag}(2^{1/2}, 1, 1, \dots, 1) R_k^{-1} \right\|_{\text{F}}^2, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \sum_j \frac{1}{[\sigma_j(V_{k,n})]^2} &= \frac{2}{n} \left\| \text{diag}(2^{-1/2}, 1, 1, \dots, 1) R_k^* \right\|_{\text{F}}^2 \\ &= \frac{2}{n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2, \end{aligned} \quad (4.6)$$

³One can also have

$$V_{k,n} V_{k,n}^T = (n/2) R_k^{-T} \text{diag}(2, 1, 1, \dots, 1) R_k^{-1}, \quad (V_{k,n} V_{k,n}^T)^{-1} = (2/n) R_k \text{diag}(2^{-1}, 1, 1, \dots, 1) R_k^T.$$

Although they are very similar to (4.3) and (4.4) but can be quite different because ω and/or τ and thus $V_{k,n}$ may possibly be complex. Whether these two identities have any use remain to be seen.

where \sum'_j means the first term is halved. (4.5) involves R_k^{-1} , making it a little hard to use without inverting R_k first. We might be better off by using $\sqrt{\sum_j [\sigma_j(V_{k,n})]^2} = \|V_{k,n}\|_F$. Nevertheless it relates the singular values to the coefficients of $T_j(x; \omega, \tau)$ in a nontrivial way.

Theorem 4.2 *Let $V_{k,n}$ have the translated Chebyshev zero nodes. Then*

$$\kappa_F(V_{k,n}) = \|V_{k,n}\|_F \sqrt{\frac{2}{n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2}.$$

Previously similar estimates (bounds) were done for $\alpha_j = t_{jn}^{\text{tr}}$ on $[\alpha, \beta]$ for the following cases (while $V_{k,n}$ in Theorem 4.2 may be complex).

- Gautschi [6]: $k = n$, $-\alpha = \beta$ or $\alpha\beta \geq 0$, and ℓ_∞ -condition number;
- Li [11]: $k = n$, $-\alpha = \beta$ or $\alpha\beta \geq 0$, and ℓ_p -condition number.

Lemma 4.1 *Let $\alpha_j = t_{jn}^{\text{tr}}$ ($1 \leq j \leq n$) on $[\alpha, \beta]$. Then $\max_j |\alpha_j| = \eta \max\{|\alpha|, |\beta|\}$, where*

$$\eta = \begin{cases} \left(\frac{1+\delta}{2} + \frac{1-\delta}{2} \cos \frac{\pi}{2n} \right) \sim 1 - \frac{1-\delta}{16n^2} \pi^2 + \mathcal{O}(n^{-4}), & \text{if } \alpha\beta \geq 0, \\ \left(\frac{1-\delta}{2} + \frac{1+\delta}{2} \cos \frac{\pi}{2n} \right) \sim 1 - \frac{1+\delta}{16n^2} \pi^2 + \mathcal{O}(n^{-4}), & \text{if } \alpha\beta < 0 \end{cases}$$

and $\delta = \min\{|\alpha|, |\beta|\} / \max\{|\alpha|, |\beta|\} \leq 1$. Consequently for $1 \leq k \leq n$

$$\eta^{k-1} \sim \begin{cases} 1 - \frac{(k-1)(1-\delta)}{16n^2} \pi^2 + \mathcal{O}((k-1)^2 n^{-4}), & \text{if } \alpha\beta \geq 0, \\ 1 - \frac{(k-1)(1+\delta)}{16n^2} \pi^2 + \mathcal{O}((k-1)^2 n^{-4}), & \text{if } \alpha\beta < 0, \end{cases}$$

Proof: The expression for η is a consequence of $t_{jn}^{\text{tr}} = \omega(t_{jn} - \tau) = \frac{\beta - \alpha}{2} \cos \frac{2j-1}{2n} \pi + \frac{\beta + \alpha}{2}$. The asymptotical expansion for η^{k-1} follows from expanding $(k-1) \ln \eta$ and then $\exp((k-1) \ln \eta)$.

■

With this lemma, we have

$$\|V_{k,n}\|_F \leq \sqrt{kn} \max\{1, \max_j |t_{jn}^{\text{tr}}|^{k-1}\} \sim \sqrt{kn} [\max\{1, |\alpha|, |\beta|\}]^{k-1}, \quad (4.7)$$

$$\|V_{k,n}\|_F \geq \max\{1, \max_j |t_{jn}^{\text{tr}}|^{k-1}\} \sim [\max\{1, |\alpha|, |\beta|\}]^{k-1}. \quad (4.8)$$

Together they imply

$$\sqrt{\sum_j [\sigma_j(V_{k,n})]^2} = \|V_{k,n}\|_F \stackrel{n}{\sim} [\max\{1, |\alpha|, |\beta|\}]^{k-1}. \quad (4.9)$$

Let us now specialize Theorem 4.2 to the cases $-\alpha = \beta$ or $\alpha\beta \geq 0$. By (2.16), (4.9), and Theorem 4.2, we have

$$\kappa_F(V_{k,n}) \stackrel{n}{\sim} [\max\{1, |\alpha|, |\beta|\}]^{k-1} \sum_{j=0}^{k-1} S_{j,1}(\omega, \tau). \quad (4.10)$$

First the case $-\alpha = \beta$. Then $\omega = \beta$ and $\tau = 0$. (4.10) and (2.12) yield

$$\begin{aligned}\kappa_{\text{F}}(V_{k,n}) &\stackrel{n}{\sim} \max\{1, \beta^{k-1}\} S_{k-1,1}(\beta, 0) \\ &\stackrel{n}{\sim} \max \left\{ \left(\frac{1}{\beta} + \sqrt{1 + \frac{1}{\beta^2}} \right)^{k-1}, \left(\beta + \sqrt{1 + \beta^2} \right)^{k-1} \right\}.\end{aligned}\quad (4.11)$$

Thus the nearly optimal conditioned $V_{k,n}$ are those with $\beta \approx 1$.

Next, consider the case $\alpha\beta \geq 0$. Without loss of generality, assume $0 \leq \alpha < \beta$. (4.10) and (2.14) yield

$$\begin{aligned}\kappa_{\text{F}}(V_{k,n}) &\stackrel{n}{\sim} \max\{1, \beta^{k-1}\} S_{k-1,1}(\omega, \tau) \\ &\geq \max\{1, \beta^{k-1}\} S_{k-1,1}(\beta/2, 1),\end{aligned}\quad (4.12)$$

and if, addition, $\alpha = 0 < \beta$,

$$\begin{aligned}\kappa_{\text{F}}(V_{k,n}) &\stackrel{n}{\sim} \max \max\{1, \beta^{k-1}\} S_{k-1,1}(\beta/2, 1) \\ &\stackrel{n}{\sim} \max \left\{ \left(\frac{2}{\beta} + 1 + 2\sqrt{\frac{1}{\beta^2} + \frac{1}{\beta}} \right)^{k-1}, \left(2 + \beta + 2\sqrt{1 + \beta} \right)^{k-1} \right\}.\end{aligned}\quad (4.13)$$

Thus the nearly optimal conditioned $V_{k,n}$ with $\alpha = 0 < \beta$ are also those with $\beta \approx 1$.

Since $\kappa_{\text{F}} \stackrel{n}{\sim} \kappa_p$, all (4.11) – (4.13) for $k = n$ can be deduced from results in [11]. They, in fact, together with Theorems 3.1 and 3.2 for $k = n$ were the foundation in [11]. Now we have similar conclusions for $V_{k,n}$ with nodes in $[\alpha, \beta]$ for all k . This is summarized in the following theorem.

Theorem 4.3 *If $-\alpha = \beta$ or $\alpha\beta \geq 0$, then subject to $(\max_j |\alpha_j|)^{n-1} \stackrel{n}{\sim} [\max\{|\alpha|, |\beta|\}]^{n-1}$,*

$$\min \kappa_{\text{F}}(V_{k,n}) \stackrel{n}{\sim} [\max\{1, |\alpha|, |\beta|\}]^{k-1} S_{k-1,1}(\omega, \tau),$$

and thus $V_{k,n}$ with $\alpha_j = t_{jn}^{\text{tr}}$ ($1 \leq j \leq n$) on $[\alpha, \beta]$ is a nearly optimally conditioned one on the interval. In particular

$$\min \kappa_{\text{F}}(V_{k,n}) \stackrel{n}{\sim} \begin{cases} \text{RHS of (4.11),} & \text{for } -\alpha = \beta, \\ \text{RHS of (4.13),} & \text{for } 0 = \alpha < \beta. \end{cases}$$

But questions such as what asymptotically optimal lower bounds and/or nearly optimally conditioned Vandermonde matrices are for interval with $-\alpha \neq \beta$, $\alpha < 0$, and $\beta > 0$ were not answered in [11]. With Theorem 4.2 here, we are one step closer as we shall explain. Answers to both questions would be firm if we could show that the right-hand sides of (3.3) and (4.10) were equivalent in the sense of $\stackrel{n}{\sim}$. We suspect this would be very much true because it would be reasonable to expect $S_{j,1}(\omega, \tau)$ to be (almost) nondecreasing as j increases, but we have no proof for now. So we formulate a conjecture as follows, which has been known true for $-\alpha = \beta$ or $\alpha\beta \geq 0$, by examining (2.12) and (2.14).

Conjecture 4.1 *For $\alpha < \beta$, ω and τ defined as in (2.18),*

$$\sum_{j=0}^{k-1} S_{j,1}(\omega, \tau) \stackrel{n}{\sim} S_{k-1,1}(\omega, \tau).$$

4.2 Extreme examples for CG and Lanczos

A key component in [12] for devising examples to achieve the sharpness of the existing error bounds for the Conjugate Gradient method (CG) [13], and the symmetric Lanczos method [9, 14] is the computation of $\min_{|u_{(1)}|=1} \|V_{k,n}^T u\|_2$ for V_n of this section. The convergence analysis of the minimal residual method (MINRES) (for $Ax = b$ with normal A) ends up with the same computation, too, except possibly complex α_j . It is proved in [12] that

$$\min_{|u_{(1)}|=1} \frac{\|V_{k,n}^T u\|_2}{\sqrt{n}} = \left[2 \sum_{j=0}^{k-1} |T_j(\tau)|^2 \right]^{-1/2} \quad (4.14)$$

for $\alpha_j = t_{jn}^{\text{tr}}$ on $[\alpha, \beta]$, as a consequence of (4.4). Equation (4.14), however, is valid for $\alpha_j = t_{jn}^{\text{tr}}$ for any $\omega \neq 0$ and τ with or without the interval $[\alpha, \beta]$ in the context. A proof can be gotten along the same line as in [12]; see also Subsection 5.2.

4.3 Bounds on individual singular values

We start with (4.4). Let the diagonal entries of its right-hand side be d_j ($1 \leq j \leq k$). Then

$$d_1 = \frac{2}{n} \sum_{i=0}^{k-1} |a_{0i}|^2, \quad d_j = \frac{2}{n} \sum_{i=j-1}^{k-1} |a_{j-1i}|^2 \quad \text{for } 2 \leq j \leq k.$$

By Schur theorem [3, p.35], $\{d_j\}_{j=1}^k$ is majorized by the eigenvalues of $(\bar{V}_{k,n} V_{k,n}^T)^{-1}$ which are $\{[\sigma_j(V_{k,n})]^{-2}\}_{j=1}^k$. Recall our default ordering (1.2) on singular values to get

$$[\sigma_1(V_{k,n})]^{-2} \geq [\sigma_2(V_{k,n})]^{-2} \geq \dots \geq [\sigma_k(V_{k,n})]^{-2}.$$

What the majorization means is if we let $\{d_j^\downarrow\}_{j=1}^k$ be the non-increasing reordering of $\{d_j\}_{j=1}^k$, i.e.,

$$d_1^\downarrow \geq d_2^\downarrow \geq \dots \geq d_k^\downarrow,$$

then

$$\sum_{j=1}^i [\sigma_j(V_{k,n})]^{-2} \geq \sum_{j=1}^i d_j^\downarrow \quad \text{for } 1 \leq i \leq k \quad (4.15)$$

which can also be equivalently stated as

$$\sum_{j=i}^k [\sigma_j(V_{k,n})]^{-2} \leq \sum_{j=i}^k d_j^\downarrow \quad \text{for } 1 \leq i \leq k. \quad (4.16)$$

Let us look at what we can draw from them. (4.16) implies

$$[\sigma_i(V_{k,n})]^{-2} \leq \sum_{j=i}^k d_j^\downarrow \quad \text{for } 1 \leq i \leq k,$$

or, equivalently

$$\sigma_i(V_{k,n}) \geq \left[\sum_{j=i}^k d_j^\downarrow \right]^{-1/2} \quad \text{for } 1 \leq i \leq k. \quad (4.17)$$

Take $i = 1$ in (4.15), combining with (4.17), to get for the smallest singular value

$$\left[\sum_{j=1}^k d_j^\downarrow \right]^{-1/2} \leq \sigma_1(V_{k,n}) \leq [d_1^\downarrow]^{-1/2}. \quad (4.18)$$

The lower bounds in (4.17) are guaranteed very sharp for $i = 1$ because of (4.18), but may not be so for $i \neq 1$. Our numerical calculations for various interval $[\alpha, \beta]$ shows that they are pretty good for the first few smallest singular values, and then deteriorate as i becomes bigger. But for $\max\{|\alpha|, |\beta|\} \approx 1$, the lower bounds are sharp for both ends of singular values (i.e., largest and smallest ones).

Exactly the same thing can be done with (4.3) upon extracting the diagonal entries of $R_k^{-*} \text{diag}(2, 1, 1, \dots, 1) R_k^{-1}$. But these diagonal entries relate to the coefficients in a much more complicated way. We shall not pursue this here.

5 V_n with Chebyshev extreme nodes

Since there is $n + 1$ extreme points for $T_n(t)$ on $[-1, 1]$, it is more convenient in terms of formula writing for us to work with V_{n+1} than V_n , and this is what we will be doing in this section. Throughout this section V_{n+1} will have nodes $\alpha_{j+1} = \tau_{jn}^{\text{tr}}$ for $0 \leq j \leq n$. Set

$$\mathbf{S}_{n+1} \stackrel{\text{def}}{=} \begin{pmatrix} T_0(\tau_{0n}) & T_1(\tau_{0n}) & T_2(\tau_{0n}) & \cdots & T_n(\tau_{0n}) \\ T_0(\tau_{1n}) & T_1(\tau_{2n}) & T_2(\tau_{2n}) & \cdots & T_n(\tau_{2n}) \\ \vdots & \vdots & \vdots & & \vdots \\ T_0(\tau_{nn}) & T_1(\tau_{nn}) & T_2(\tau_{nn}) & \cdots & T_n(\tau_{nn}) \end{pmatrix}. \quad (5.1)$$

Then $V_{n+1}^T R_{n+1} = \mathbf{S}_{n+1}$. \mathbf{S}_{n+1} is always real, while V_{n+1} and R_{n+1} may not. We now compute

$$\Upsilon_{n+1} \stackrel{\text{def}}{=} \mathbf{S}_{n+1}^T \Omega_{n+1} \mathbf{S}_{n+1}, \quad (5.2)$$

where

$$\Omega_{n+1} = \text{diag}(2^{-1}, 1, 1, \dots, 1, 2^{-1}) \in \mathbb{R}^{(n+1) \times (n+1)}. \quad (5.3)$$

To this end, we notice

$$(\mathbf{S}_{n+1})_{(i+1, j+1)} = T_j(\tau_{in}) = \cos j\vartheta_{in} = \cos \frac{ji}{n}\pi,$$

and therefore for $0 \leq i, j \leq n$

$$(\mathbf{S}_{n+1}^T \Omega_{n+1} \mathbf{S}_{n+1})_{(i+1, j+1)} = \sum_{k=0}^n {}'' (\mathbf{S}_{n+1}^T)_{(i+1, k+1)} (\mathbf{S}_{n+1})_{(k+1, j+1)}$$

$$\begin{aligned}
&= -\frac{T_i(\tau_{0n})T_j(\tau_{0n}) + T_i(\tau_{nn})T_j(\tau_{nn})}{2} + \sum_{k=0}^n T_i(\tau_{kn})T_j(\tau_{kn}) \\
&= -\frac{1 + (-1)^{i+j}}{2} + \sum_{k=0}^n \cos i\vartheta_{kn} \cos j\vartheta_{kn} \\
&= -\frac{1 + (-1)^{i+j}}{2} + \frac{1}{2} \sum_{k=0}^n \cos(i+j)\vartheta_{kn} + \frac{1}{2} \sum_{k=0}^n \cos(i-j)\vartheta_{kn}, \tag{5.4}
\end{aligned}$$

where \sum_j'' means the first and last terms are halved. We now compute $\sum_{k=0}^n \cos \ell\vartheta_{kn}$, where ℓ is an integer. We claim that

$$\sum_{k=0}^n \cos \ell\vartheta_{kn} = \begin{cases} n+1, & \text{if } \ell = 2mn \text{ for some integer } m, \\ 0, & \text{if } \ell \text{ is odd,} \\ 1, & \text{if } \ell \text{ is even, but } \ell \neq 2mn \text{ for any integer } m. \end{cases} \tag{5.5}$$

Since $\ell\vartheta_{kn} = (\ell k/n)\pi$, the case $\ell = 2mn$ is clear. Assume that $\ell \neq 2mn$ for any integer m , and then $\cos \phi \neq 1$, where $\phi = \ell\pi/n$. Denote $\iota = \sqrt{-1}$. We have

$$\begin{aligned}
2 \sum_{k=0}^n \cos \ell\vartheta_{kn} &= 2 \sum_{k=0}^n \cos k\phi = \sum_{k=0}^n [e^{\iota k\phi} + e^{-\iota k\phi}] \\
&= \sum_{k=0}^n [e^{\iota\phi}]^k + \sum_{k=0}^n [e^{-\iota\phi}]^k \\
&= \frac{1 - [e^{\iota\phi}]^{n+1}}{1 - e^{\iota\phi}} + \frac{1 - [e^{-\iota\phi}]^{n+1}}{1 - e^{-\iota\phi}} \\
&= \frac{1 - e^{\iota(n+1)\phi}}{1 - e^{\iota\phi}} + \frac{1 - e^{-\iota(n+1)\phi}}{1 - e^{-\iota\phi}} \\
&= \frac{1 + \cos n\phi - \cos \phi - \cos(n+1)\phi}{1 - \cos \phi} \\
&= 1 + (-1)^\ell
\end{aligned}$$

upon noticing $\cos n\phi = \cos \ell\pi = (-1)^\ell$ and $\cos(n+1)\phi = \cos(\ell\pi + \phi) = (-1)^\ell \cos \phi$. (5.5) is proved. Consider now $\ell = i \pm j$, and $0 \leq i, j \leq n$. Since $0 \leq i+j \leq 2n$ and $-n \leq i-j \leq n$, for some integer m

$$\begin{aligned}
i+j = 2mn &\Leftrightarrow i=j=0, \quad \text{or } i=j=n; \\
i-j = 2mn &\Leftrightarrow i=j.
\end{aligned}$$

It follows from (5.4) and (5.5) that

$$\Upsilon_{n+1} = \frac{n}{2} \text{diag}(2, 1, 1, \dots, 1, 2). \tag{5.6}$$

Equation (5.1) yields $V_{n+1}^T = \mathbf{S}_{n+1} R_{n+1}^{-1}$. Extracting the first k columns from the both sides yields the following theorem.

Theorem 5.1 *Let V_{n+1} have the translated Chebyshev extreme nodes $\alpha_j = \tau_{jn}^{\text{tr}}$ ($1 \leq j \leq n$), and let upper triangular R_k be defined as in (2.11) and \mathbf{S}_{n+1} as in (5.1). Then $V_{k,n+1}^T = (\mathbf{S}_{n+1})_{(:,1:k)} R_k^{-1}$.*

5.1 Condition number $\kappa_F(V_{k,n+1})$

A rough bound for Υ_{n+1} is⁴

$$\frac{n}{2}I_{n+1} \leq \Upsilon_{n+1} \leq n I_{n+1} \quad (5.7)$$

which is probably good enough for most occasions. (5.7) implies immediately

$$\frac{n}{2}I_k \leq (\Upsilon_{n+1})_{(1:k,1:k)} \leq n I_k, \quad \frac{1}{n}I_k \leq [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} \leq \frac{2}{n}I_k \quad (5.8)$$

By Theorem 5.1, we have⁵

$$\begin{aligned} \bar{V}_{k,n+1}\Omega_{n+1}V_{k,n+1}^T &= R_k^{-*} [(\mathbf{S}_{n+1})_{(:,1:k)}]^T \Omega_{n+1} (\mathbf{S}_{n+1})_{(:,1:k)} R_k^{-1} \\ &= R_k^{-*} (\mathbf{S}_{n+1}^T \Omega_{n+1} \mathbf{S}_{n+1})_{(1:k,1:k)} R_k^{-1} \\ &= R_k^{-*} (\Upsilon_{n+1})_{(1:k,1:k)} R_k^{-1}, \end{aligned} \quad (5.9)$$

$$(\bar{V}_{k,n+1}\Omega_{n+1}V_{k,n+1}^T)^{-1} = R_k [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} R_k^*, \quad (5.10)$$

where $\bar{V}_{k,n+1}$ is its complex conjugate. Since $(1/2)I_{n+1} \leq \Omega_{n+1} \leq I_{n+1}$ and (5.8),

$$\begin{aligned} \frac{1}{2}\bar{V}_{k,n+1}V_{k,n+1}^T &\leq \bar{V}_{k,n+1}\Omega_{n+1}V_{k,n+1}^T \leq \bar{V}_{k,n+1}V_{k,n+1}^T, \\ \frac{n}{2}R_k^{-*} R_k^{-1} &R_k^{-*} (\Upsilon_{n+1})_{(1:k,1:k)} R_k^{-1} \leq nR_k^{-*} R_k^{-1}, \\ (\bar{V}_{k,n+1}V_{k,n+1}^T)^{-1} &\leq (\bar{V}_{k,n+1}\Omega_{n+1}V_{k,n+1}^T)^{-1} \leq 2(\bar{V}_{k,n+1}V_{k,n+1}^T)^{-1}, \\ \frac{1}{n}R_k R_k^* &\leq R_k [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} R_k^* \leq \frac{2}{n}R_k R_k^*. \end{aligned}$$

Consequently,

$$\frac{n}{2} \|R_k^{-1}\|_F^2 \leq \sum_j [\sigma_j(V_{k,n+1})]^2 \leq 2n \|R_k^{-1}\|_F^2, \quad (5.11)$$

$$\frac{1}{2n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2 \leq \sum_j \frac{1}{[\sigma_j(V_{k,n+1})]^2} \leq \frac{2}{n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2. \quad (5.12)$$

Theorem 5.2 *Let $V_{k,n+1}$ have the translated Chebyshev extreme nodes τ_{jn}^{tr} . Then*

$$\|V_{k,n+1}\|_F \sqrt{\frac{1}{2n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2} \leq \kappa_F(V_{k,n+1}) \leq \|V_{k,n+1}\|_F \sqrt{\frac{2}{n} \sum_{j=0}^{k-1} [S_{j,2}(\omega, \tau)]^2}.$$

As what we did for Theorem 4.2, we may specialize this theorem onto interval $[\alpha, \beta]$ with the cases $-\alpha = \beta$ or $\alpha\beta \geq 0$ in an almost the same way to conclude that (4.10) – (4.13) remain true with $V_{k,n}$ there replaced by $V_{k,n+1}$ here. Detail is omitted.

⁴ $X \leq Y$ for two Hermitian matrices means that $Y - X$ is positive semidefinite.

⁵Again one can also have

$$V_{k,n}\Omega_{n+1}V_{k,n}^T = (n/2)R_k^{-*} (\Upsilon_{n+1})_{(1:k,1:k)} R_k^{-1}, \quad (V_{k,n}\Omega_{n+1}V_{k,n}^T)^{-1} = (2/n)R_k (\Upsilon_{n+1})_{(1:k,1:k)}^{-1} R_k^T,$$

similar to (5.9) and (5.10).

5.2 Extreme examples for CG and Lanczos

V_{n+1} here provides another example to achieve the sharpness of the existing error bounds for CG, MINRES, and the symmetric Lanczos method. We shall just compute (or estimate) $\min_{|u_{(1)}|=1} \|\Omega_{n+1}^{1/2} V_{k,n+1}^T u\|_2$ and the rest would be straightforward minor modifications to what in [12] and thus will be omitted.

Our inequalities (5.13) and (5.15) below are more general than what CG needs. They are valid, regardless whether all nodes are positive or not. It follows from a theorem in [12] that

$$\min_{|u_{(1)}|=1} \|\Omega_{n+1}^{1/2} V_{k,n+1}^T u\|_2 = [e_1^T (\bar{V}_{k,n+1} \Omega_{n+1} V_{k,n+1}^T)^{-1} e_1]^{-1/2}.$$

By (5.10), we have

$$e_1^T (V_{k,n+1} \Omega_{n+1} V_{k,n+1}^T)^{-1} e_1 = y^* [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} y,$$

where $y \in \mathbb{C}^k$ and $y_{(j+1)} = a_{0j} = T_j(\tau)$. Immediately with (5.8), we get

$$\min_{|u_{(1)}|=1} \frac{\|\Omega_{n+1}^{1/2} V_{k,n+1}^T u\|_2}{\sqrt{n}} = \frac{1}{\sqrt{2\Gamma_{k,\tau}}}, \quad (5.13)$$

where

$$\Gamma_{k,\tau} \stackrel{\text{def}}{=} \begin{cases} \sum_{j=0}^{k-1} |T_j(\tau)|^2, & \text{for } k \leq n, \\ \sum_{j=0}^n |T_j(\tau)|^2, & \text{for } k = n+1. \end{cases} \quad (5.14)$$

Now for any given $g \in \mathbb{C}^{n+1}$, we have $\text{diag}(g) V_{k,n+1}^T u = \text{diag}(g) \Omega_{n+1}^{-1/2} \Omega_{n+1}^{1/2} V_{k,n+1}^T u$ and thus

$$\frac{\sqrt{n} \min_j |g_{(j)}|}{\|g\|_2} \frac{1}{\sqrt{2\Gamma_{k,\tau}}} \leq \min_{|u_{(1)}|=1} \frac{\|\text{diag}(g) V_{k,n+1}^T u\|_2}{\|g\|_2} \leq \frac{\sqrt{n} \max_j |g_{(j)}|}{\|g\|_2} \sqrt{\frac{1}{\Gamma_{k,\tau}}}. \quad (5.15)$$

(5.13) and (5.15) are valid for $\alpha_{j+1} = \tau_{jn}^{\text{tr}}$ for $0 \leq j \leq n$ for any $\omega \neq 0$ and τ with or without the interval $[\alpha, \beta]$ in the context.

Meinardus [13] showed that if $0 < \alpha < \beta$, and $g \in \mathbb{C}^{n+1}$ with

$$g_{(j+1)} = \begin{cases} \sqrt{1/\tau_{jn}^{\text{tr}}}, & \text{for } j \in \{0, n\}, \\ \sqrt{2/\tau_{jn}^{\text{tr}}}, & \text{for } 1 \leq j \leq n-1, \end{cases} \quad (5.16)$$

then

$$\min_{|u_{(1)}|=1} \frac{\|\text{diag}(g) V_{n+1}^T u\|_2}{\|g\|_2} = 2 [\Delta_\delta^n + \Delta_\delta^{-n}]^{-1}, \quad (5.17)$$

where $\delta = \alpha/\beta$ and Δ_δ is defined by (2.4). This is not exactly the way Meinardus [13] stated, but an equivalent form. Note that $V_{n+1} \equiv V_{k,n+1}$ for $k = n+1$; so the left-hand side of (5.17) is nothing but

$$\min_{|u_{(1)}|=1} \frac{\|\text{diag}(g) V_{k,n+1}^T u\|_2}{\|g\|_2} \quad \text{for } k = n+1 \text{ and } g \text{ as in (5.16)}.$$

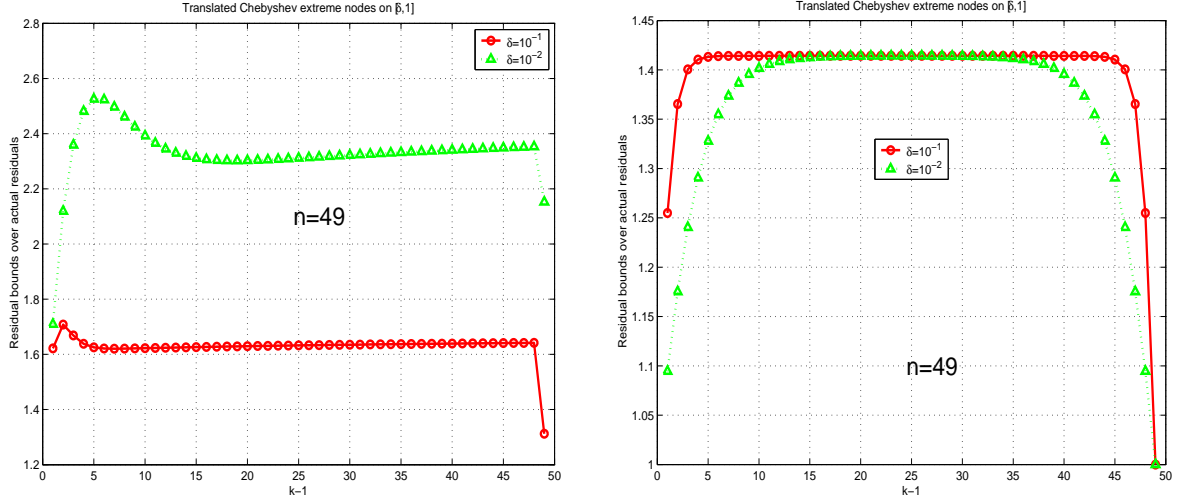


Figure 5.1: Ratios of $2 \left[\Delta_{\delta}^{k-1} + \Delta_{\delta}^{-(k-1)} \right]^{-1}$ over $\min_{|u_{(1)}|=1} \frac{\|\text{diag}(g)V_{k,n+1}^T u\|_2}{\|g\|_2}$, where $\alpha_{j+1} = \tau_{jn}^{\text{tr}}$ on $[\delta, 1]$. **Left:** $g_{(j)} = 1$ for $1 \leq j \leq n+1$; **Right:** g defined by (5.16).

Numerical tests indicate that for $1 \leq k \leq n$

$$\min_{|u_{(1)}|=1} \frac{\|\text{diag}(g)V_{k,n+1}^T u\|_2}{\|g\|_2} < 2 \left[\Delta_{\delta}^{k-1} + \Delta_{\delta}^{-(k-1)} \right]^{-1}.$$

Our (5.15) can lead to a lower and a upper bound (for all k) that differ by at most a factor of $2\sqrt{2\beta/\alpha}$ because $\min_j |g_{(j)}| \geq 1/\sqrt{\beta}$ and $\max_j |g_{(j)}| \geq \sqrt{2/\alpha}$. For comparison purpose, let cite the following restatement of a result also due to Meinardus [13]: *If all $\alpha_j \in [\alpha, \beta]$ (not necessarily the translated Chebyshev extreme nodes, of course) and $0 < \alpha$, then for any $g \in \mathbb{C}^{n+1}$,*

$$\min_{|u_{(1)}|=1} \frac{\|\text{diag}(g)V_{k,n+1}^T u\|_2}{\|g\|_2} \leq 2 \left[\Delta_{\delta}^{k-1} + \Delta_{\delta}^{-(k-1)} \right]^{-1} \quad (5.18)$$

for all $1 \leq k \leq n+1$. Figure 5.1 plots the ratio of the right-hand side of (5.18) over its left-hand side for $\alpha_{j+1} = \tau_{jn}^{\text{tr}}$ on $[\delta, 1]$ with two different g . It is interesting to notice a sudden drop at $k-1 = n$ for g being the vector of all ones, and for g as in (5.16), the ratio is one at $k-1 = n$ guaranteed by (5.17) and for all other k the ratios appear no bigger than $\sqrt{2}$. This has recently been confirmed by the author in a report in preparation.

5.3 Bounds on individual singular values

Let the diagonal entries of $R_k [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} R_k^*$ be

$$\delta_j = e_j^T R_k [(\Upsilon_{n+1})_{(1:k,1:k)}]^{-1} R_k^* e_j \quad \text{for } 1 \leq j \leq k.$$

and let the nonincreasing re-ordering of δ_j 's be δ_j^{\downarrow} 's, i.e.,

$$\delta_1^{\downarrow} \geq \delta_2^{\downarrow} \geq \dots \geq \delta_k^{\downarrow}.$$

Equation (5.10) implies, by Schur theorem [3, p.35],

$$\sum_{j=1}^i [\sigma_j(V_{k,n+1})]^{-2} \geq \frac{1}{2} \sum_{j=1}^i \delta_j^\downarrow \quad \text{for } 1 \leq i \leq k, \quad (5.19)$$

$$\sum_{j=i}^k [\sigma_j(V_{k,n+1})]^{-2} \leq \sum_{j=i}^k \delta_j^\downarrow \quad \text{for } 1 \leq i \leq k. \quad (5.20)$$

Now (5.20) implies

$$[\sigma_i(V_{k,n+1})]^{-2} \leq \sum_{j=i}^k \delta_j^\downarrow \quad \text{for } 1 \leq i \leq k,$$

or, equivalently

$$\sigma_i(V_{k,n+1}) \geq \left[\sum_{j=i}^k \delta_j^\downarrow \right]^{-1/2} \quad \text{for } 1 \leq i \leq k. \quad (5.21)$$

Take $i = 1$ in (5.19), combining with (5.21), to get for the smallest singular value

$$\left[\sum_{j=1}^k \delta_j^\downarrow \right]^{-1/2} \leq \sigma_1(V_{k,n+1}) \leq \sqrt{2} \left[\delta_1^\downarrow \right]^{-1/2}. \quad (5.22)$$

Our comments at the end of Subsection 4.3 apply here, too.

6 V_n with other orthogonal polynomial zero nodes

Part of the material in Section 4 can naturally be extended to V_n whose nodes are the zeros of the n th translated orthogonal polynomial from any orthogonal polynomial system. We shall outline the detail here. Let $p_j(t)$, $j = 0, 1, 2, \dots$, denote a sequence of *normalized* orthogonal polynomial with respect to some weight function $w(t)$ on an interval which may be open, half open, or closed. For a list of well-known orthogonal polynomials such as T_n we have been dealing with so far, Legendre polynomials, and etc, the reader is referred to [4, p.57] or any books on orthogonal polynomials. Orthogonal polynomials have many beautiful properties. Useful to us here are that $p_n(t)$ is guaranteed to have exact n distinct zeros⁶ t_{jn} , $j = 1, 2, \dots, n$, in the interval, and [7]

$$\sum_{j=1}^n \lambda_{jn} p_r(t_{jn}) p_s(t_{jn}) = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } 0 \leq r, s \leq n-1 \quad (6.1)$$

as the result of the fact that the Gaussian quadrature formula

$$\int \phi(t) w(t) dt \approx \sum_{j=1}^n \lambda_{jn} \phi(t_{jn})$$

⁶Up to now, t_{jn} is reserved for the zeros of the n th Chebyshev polynomial of the first kind. It, along with other previously reserved notation, will be reassigned in this section.

is exact for all polynomials of degree no higher than $2n - 1$ and $\int p_r(t)p_s(t)w(t)dt = 0$ for $r \neq s$ and 1 for $r = s$, where the integral is taken over the support of $w(t)$, λ_{jn} are Christoffel numbers for p_n . For numerical values of the nodes and Christoffel numbers for various orthogonal polynomials, the reader is referred to [1].

Given $\omega \neq 0$ and τ , define translated orthogonal polynomials by

$$\begin{aligned} p_n(x; \omega, \tau) &\stackrel{\text{def}}{=} p_n(x/\omega + \tau), \\ &= a_{nn}x^n + a_{n-1,n}x^{n-1} + \cdots + a_{1n}x + a_{0n}, \end{aligned}$$

whose zeros are

$$t_{jn}^{\text{tr}} = \omega(t_{jn} - \tau), \quad 1 \leq j \leq n.$$

Let V_n be with $\alpha_j = t_{jn}^{\text{tr}}$ and set R_n as in (2.11) but with a_{ij} here. Set

$$\mathbf{P}_n \stackrel{\text{def}}{=} \begin{pmatrix} p_0(t_{1n}) & p_1(t_{1n}) & p_2(t_{1n}) & \cdots & p_{n-1}(t_{1n}) \\ p_0(t_{2n}) & p_1(t_{2n}) & p_2(t_{2n}) & \cdots & p_{n-1}(t_{2n}) \\ \vdots & \vdots & \vdots & & \vdots \\ p_0(t_{nn}) & p_1(t_{nn}) & p_2(t_{nn}) & \cdots & p_{n-1}(t_{nn}) \end{pmatrix}. \quad (6.2)$$

Then $V_n^T R_n = \mathbf{P}_n$, and

$$\mathbf{P}_n^T \Lambda_n \mathbf{P}_n = I_n, \quad (6.3)$$

where $\Lambda_n = (\lambda_{1n}, \lambda_{2n}, \dots, \lambda_{nn})$. Therefore $V_n^T = \mathbf{P}_n R_n^{-1}$. Extracting the first k columns from both sides of $V_n^T = \mathbf{P}_n R_n^{-1}$ yields $V_{k,n}^T = (\mathbf{P}_n)_{(:,1:k)} R_k^{-1}$, similar to Theorem 4.1.

Let $\Lambda_{k,n} = (\Lambda_n)_{(1:k,1:k)}$. We have

$$\begin{aligned} \bar{V}_{k,n} \Lambda_{k,n} V_{k,n}^T &= R_k^{-*} [(\mathbf{P}_n)_{(:,1:k)}]^T \Lambda_{k,n} (\mathbf{P}_n)_{(:,1:k)} R_k^{-1} \\ &= R_k^{-*} (\mathbf{P}_n^T \Lambda_n \mathbf{P}_n)_{(1:k,1:k)} R_k^{-1} \\ &= R_k^{-*} R_k^{-1}, \end{aligned} \quad (6.4)$$

$$(\bar{V}_{k,n} \Lambda_{k,n} V_{k,n}^T)^{-1} = R_k R_k^*. \quad (6.5)$$

Upon noticing $\min_j \lambda_{jn} I_k \leq \Lambda_{k,n} \leq \max_j \lambda_{jn} I_k$, we have

$$\frac{1}{\max_j \lambda_{jn}} \|R_k^{-1}\|_{\text{F}}^2 \leq \sum_j [\sigma_j(V_{k,n})]^2 \leq \frac{1}{\min_j \lambda_{jn}} \|R_k^{-1}\|_{\text{F}}^2, \quad (6.6)$$

$$\left(\min_j \lambda_{jn}\right) \Phi_k \leq \sum_j \frac{1}{[\sigma_j(V_{k,n})]^2} \leq \left(\max_j \lambda_{jn}\right) \Phi_k. \quad (6.7)$$

where

$$\Phi_k \stackrel{\text{def}}{=} \|R_k\|_{\text{F}}^2 = \sum_{j=0}^{k-1} \sum_{i=0}^j a_{ij}^2.$$

Now bounds on $\kappa_{\text{F}}(V_{k,n})$ can be easily obtained in terms of a_{ij} . For CG or MINRES residuals, we have exactly

$$\min_{|u_{(1)}|=1} \|\text{diag}(g) V_{k,n}^T u\|_2 = \left[\sum_{j=0}^{k-1} |p_j(\tau)|^2 \right]^{-1/2}, \quad (6.8)$$

where $g = (\sqrt{\lambda_{1n}}, \sqrt{\lambda_{2n}}, \dots, \sqrt{\lambda_{nn}})^T$. We may also get bounds on individual singular value $\sigma_j(V_{k,n})$, following the lines of previous sections. For similarity reason, detail is omitted.

7 Conclusions

Vandermonde matrices with translated Chebyshev zero and extreme nodes are shown to have various interesting properties, derived from simple QR or QR-like decompositions. These decompositions allow us to obtain the behavior of their condition numbers, and bounds on individual singular value.

This simple QR-like decomposition for V_n with translated Chebyshev zero nodes is shared by a much larger class of V_n , i.e., those with nodes being translated zeros of any orthogonal polynomials. Consequently we can get about the same things as we did for V_n with translated Chebyshev zero nodes.

There are two immediate applications of studying Vandermonde matrices with translated Chebyshev nodes. The first one is to establish asymptotically optimal lower bounds on condition numbers of real *rectangular* Vandermonde matrices and nearly optimal conditioned real *rectangular* Vandermonde matrices on a given interval. Previously similar results were obtained for real *square* Vandermonde matrices in [2, 11]. The second application is its implication to the convergence of the Conjugate Gradient method for positive definite linear systems and the symmetric Lanczos algorithm for symmetric eigenvalue problems. It is observed that superlinear convergence [8, 15, 16] often occur for the two methods; while existing error bounds [9, 13, 14] only guaranteed linear convergence. But little study has been done as to the sharpness of these error bounds. Results in this paper can be used to construct examples as in [12] for which errors of approximations by both methods are comparable to the existing error bounds at all iteration steps.

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