Let $B$ be an $m \times n$ ($m \geq n$) complex (or real) matrix. It is known that there is a unique polar decomposition $B = QH$, where $Q^*Q = I$, the $n \times n$ identity matrix, and $H$ is positive definite, provided $B$ has full column rank. If $B$ is perturbed to $\tilde{B}$, how do the polar factors $Q$ and $H$ change? This question has been investigated quite extensively, but most work so far was on how the perturbation changed the unitary polar factor $Q$, and very little on the positive polar factor $H$, except $\|H - \tilde{H}\|_F \leq \sqrt{2}\|B - \tilde{B}\|_F$ in the Frobenius norm, due to F. Kittaneh (Comm. Math. Phys., 104 (1986), pp. 307–310), where $Q$ and $\tilde{H}$ are the corresponding polar factors of $\tilde{B}$. While this inequality of Kittaneh shows that $H$ is always well-behaved under perturbations, it does not tell much about smaller entries of $H$ in the case when $H$’s entries vary a great deal in magnitudes. This paper is intended to fill the gap by addressing the variations of $H$ for a graded matrix $B = GS$, where $S$ is a scaling matrix and usually diagonal (but may not be). The elements of $S$ can vary wildly, while $G$ is well-conditioned. In cases as such, the magnitudes of $H$’s entries indeed often vary a lot and thus any bound on $\|H - \tilde{H}\|_F$ means little, if any thing, to the accuracy of $\tilde{H}$’s smaller entries. This paper proposes a new way to measure the errors in the $H$ factor via bounding the scaled difference $(\tilde{H} - H)S^{-1}$, as well as how to accurately compute the factor when $S$ is diagonal. Numerical examples are presented.

The results are also extended to the matrix square root of a graded positive definite matrix.