Elliptic Genera

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The name of *elliptic genus* has been given to various multiplicative cobordism invariants taking values in a ring of modular forms. What follows is an attempt to present the simplest case—level 2 genera in characteristic $\neq 2$—in a unified way. We find it convenient to use N. Katz's approach to modular forms (cf. [7]) and view a modular form as a function of elliptic curves with a chosen invariant differential. A similar approach to elliptic genera was used by Jens Franke [3].

JACOBI FUNCTIONS. – Let K be any perfect field of characteristic $\neq 2$ and fix an algebraic closure $\bar{K}$ of K. Consider a triple (E, ω, α) consisting of

E	an elliptic curve over K, i.e. a smooth curve of genus 1 with a specified K-rational basepoint O,
ω	an invariant K-rational differential, and
α	a K-rational primitive 2-division point.

Following Igusa [6] (up to a point), we associate to these data two functions, x and y, as follows.

The set $E_4 \subset E(\bar{K})$ of 4-division points on E can be described as follows. There are four 2-division points t (α is one of them), four primitive 4-division points r such that $2r = \alpha$, and eight primitive 4-division points s such that $2s \neq \alpha$. Consider the degree 0 divisor $D = \sum(t) - \sum(r)$. Since $\sum t - \sum r = 0$ in E, and Galois symmetries transform D into itself, Abel's theorem (cf., for example, [11], III.3.5.1) implies that there is a function $x \in K(E)^{\times}$, uniquely defined up to a multiplicative constant, such that $\mathrm{div}(x) = D$.

The function x is odd, satisfies $x(u + \alpha) \equiv x(u)$, and undergoes sign changes under the two other translations of exact order 2. Moreover, if

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$r \in E_4$ satisfies $2r = \alpha$, the translation by r transforms x into Cx^{-1} for some non-zero constant C. This constant depends on the choice of r but only up to sign. It follows that $x^2(u+r)x^2(u)$ does not depend on the choice of r. We call this constant ε^{-1}, i.e.

$$\varepsilon \equiv x^{-2}(u+r)x^{-2}(u).$$

We also define

$$\delta = \tfrac{1}{8}\textstyle\sum x^{-2}(s)$$

(the summation is over the primitive 4-division points s such that $2s \neq \alpha$). If a is one of the values of $x(s)$, the other values are $\pm a$ and $\pm\varepsilon^{-1/2}a^{-1}$, each taken twice. It follows that

$$\delta = \tfrac{1}{2}(a^{-2} + \varepsilon a^2)$$

and

$$\prod(X - x(s)) = \varepsilon^{-2}(1 - 2\delta X^2 + \varepsilon X^4)^2 = \varepsilon^{-2}R(X)^2.$$

It is now easy to see that

$$\operatorname{div}(R(x)) = 2\Big(\sum(s) - 2\sum(r)\Big).$$

Using once more Abel's theorem, we see that there is a unique $y \in K(E)^\times$ such that $\operatorname{div}(y) = \sum(s) - 2\sum(r)$, and $y(O) = 1$. Since $x(O) = 0$, we have $y^2 = R(x)$.

The differential dx has four double poles r. Also, it is easy to see that s is a double zero of $x - x(s)$, hence a simple zero of dx. We conclude that

$$\operatorname{div}(dx) = \sum(s) - 2\sum(r) = \operatorname{div}(y).$$

and that dx/y is an invariant differential on E.

A slight modification of the argument given in [6] shows that the Jacobi functions satisfy the equation

$$x(u+v)(1 - \varepsilon x^2(u)x^2(v)) = x(u)y(v) + x(v)y(u),$$

known as the Euler addition formula. Accordingly, we define the *Euler formal group law* $F(U,V) \in K[[U,V]]$ by

$$F(U,V) = \frac{U\sqrt{R(V)} + V\sqrt{R(U)}}{1 - \varepsilon U^2V^2}.$$

Notice that since $\operatorname{char} K \neq 2$, $F(U,V)$ is defined over K.

The Elliptic Genus. – At this point, *we normalize x over K by requiring that $dx/y = \omega$* (the given invariant differential). All the objects $x, y, \delta, \varepsilon$, and $F(U,V)$ are now completely determined by the initial data. Replacing ω by $\lambda\omega$ ($\lambda \in K^\times$) yields:

$$x \rightsquigarrow \lambda x, \quad y \rightsquigarrow y, \quad \delta \rightsquigarrow \lambda^{-2}\delta, \quad \varepsilon \rightsquigarrow \lambda^{-4}\varepsilon, \quad F(U,V) \rightsquigarrow \lambda F(\lambda^{-1}U, \lambda^{-1}V). \tag{1}$$

As any formal group law, $F(U,V)$ is classified by a unique ring homomorphism

$$\psi : \Omega_*^{\mathrm{U}} \longrightarrow K$$

from the complex cobordism ring. Since $F(-U,-V) = -F(U,V)$, it is easy to see that ψ uniquely factors through a ring homomorphism

$$\varphi : \Omega_*^{\mathrm{SO}} \longrightarrow K$$

from the oriented cobordism ring. By definition, φ is the *level* 2 *elliptic genus.*

Suppose now that $\operatorname{char} K = 0$. Define a local parameter z near O so that $z(O) = 0$ and $dz = \omega$. Then x can be expanded into a formal power series $x(z) \in K[[z]]$ which clearly satisfies $x(z) = z + o(z)$ and $x(-z) = -x(z)$. In this case, the elliptic genus can be defined as the Hirzebruch genus (cf. [4] or [5]) corresponding to the series $P(z) = z/x(z)$. Since we have $d\,x(z)/dz = y(z)$, the logarithm $g(z)$ of this elliptic genus is given by the elliptic integral

$$g(z) = \int_0^z \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}, \tag{2}$$

which brings us to the original definition given in [9].

Modularity. – For any closed oriented manifold M of dimension $4k$, $\varphi(M)$ is a function of the triple (E, ω, α). As easily follows from (1), multiplying ω by λ results in multiplying $\varphi(M)$ by λ^{-2k}. Also, $\varphi(M)$ depends only on the isomorphism class of the triple (E, ω, α) and commutes with arbitrary extensions of the scalar field K. In the terminology of N. Katz ([7]) (adapted here to modular forms over fields), $\varphi(M)$ is a modular form of level 2 and weight $2k$. Let $\mathcal{M}_*$ be the graded ring of all such modular forms. We have $\varphi(M) \in \mathcal{M}_{2k}$, $\delta \in \mathcal{M}_2$, $\varepsilon \in \mathcal{M}_4$. Morover, one can prove that $\mathcal{M}_* \cong \mathbf{Z}[\frac{1}{2}, \delta, \varepsilon]$. If we identify these two isomorphic rings, the elliptic genus becomes the Hirzebruch genus

$$\varphi : \Omega_*^{\mathrm{SO}} \longrightarrow \mathbf{Z}[\tfrac{1}{2}, \delta, \varepsilon]$$

with logarithm given by the formal integral (2).

Integrality. – Consider

$$\tilde{\varphi} : \Omega_*^{\mathrm{Spin}} \longrightarrow \mathcal{M}_*$$

— the composition of φ with the forgetful homomorphism $\Omega_*^{\mathrm{Spin}} \longrightarrow \Omega_*^{\mathrm{SO}}$. As is shown in [2],

$$\tilde{\varphi}(\Omega_*^{\mathrm{Spin}}) = \mathbf{Z}[8\delta, \varepsilon].$$

The ring $\mathbf{Z}[8\delta, \varepsilon]$ agrees with the ring $\mathcal{M}_*(\mathbf{Z})$ of modular forms *over* $\mathbf{Z}$. Thus we have the following

Theorem 1. *If M is a Spin-manifold of dimension $4k$, then $\varphi(M) \in \mathcal{M}_{2k}(\mathbf{Z})$.*

Example: The Tate Curve. – Let K be a local field, complete with respect to a discrete valuation v, and let $q \in K^\times$ be any element satisfying $v(q) < 0$. Consider $E = K^\times / q^{2\mathbf{Z}}$. It is well-known (cf. [11], §C.14) that E can be identified with the elliptic curve (known as *the Tate curve*)

$$E_{q^2} : Y^2 + XY = X^3 + a_4 X + a_6,$$

where

$$a_4 = \sum_{m \geq 1} \left(-5m^3\right) \frac{q^{2m}}{1 - q^{2m}},$$

$$a_6 = \sum_{m \geq 1} \left(-\frac{5m^3 + 7m^5}{12}\right) \frac{q^{2m}}{1 - q^{2m}}.$$

We will treat E as an elliptic curve over K with $O = 1$ and fix the invariant differential $\omega = du/u$ ($u \in K^\times$) on E (ω corresponds to the differential $\omega_{\mathrm{can}} = dX/(2Y + X)$ on the Tate curve). E has three K-rational primitive 2-division points $-1, q$, and $-q$. To describe the corresponding Jacobi function x, consider the theta-function

$$\Theta(u) = (1 - u^{-2}) \prod_{n > 0} (1 - q^{2n} u^{-2})(1 - q^{2n} u^2).$$

This is a “holomorphic” function on $K^\times$ with simple zeroes at points of $\pm q^{\mathbf{Z}}$ (cf. [10] for a justification of this terminology), satisfying

$$\Theta(-u) = \Theta(u), \quad \Theta(q^{-1}u) = -u^2\Theta(u).$$

Consider the case where $\alpha = -1$. Let $i \in \bar{K}$ be any square root of -1, and let

$$f(u) = \frac{\Theta(u)}{\Theta(iu)} = \frac{u^2-1}{u^2+1}\prod_{n>0}\frac{(1-q^{2n}u^{-2})(1-q^{2n}u^2)}{(1+q^{2n}u^{-2})(1+q^{2n}u^2)} \tag{3}$$

f is a meromorphic function on E satisfying $f(iu) = 1/f(u)$ and

$$\operatorname{div}(f) = (1) + (-1) + (q) + (-q) - (i) - (-i) - (iq) - (-iq),$$

i.e. f is a multiple of the Jacobi function x of $(E, \omega, -1)$.

Notice now that the normalization condition $du/u = dx/y$ can be written $y(u) = ux'(u)$, where $x'(u)$ is the derivative with respect to u. Since $y(1) = 0$, we have $x'(1) = 1$. Differentiating (3), we get

$$f'(1) = \prod_{n>0}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2,$$

$$x(u) = \frac{u^2-1}{u^2+1}\prod_{n>0}\frac{(1-q^{2n}u^{-2})(1-q^{2n}u^2)(1+q^{2n})^2}{(1+q^{2n}u^{-2})(1+q^{2n}u^2)(1-q^{2n})^2}.$$

and

$$\varepsilon = \prod_{n>0}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^8.$$

Finally, if $\operatorname{char} K = 0$, the function $z = \log u$ satisfies $dz = du/u$. It follows that the generating series $P(z) = z/x(z)$ is given by

$$P(z) = \frac{z}{\tanh z}\prod_{n>0}\frac{(1+q^{2n}e^{-2z})(1+q^{2n}e^{2z})(1-q^{2n})^2}{(1-q^{2n}e^{-2z})(1-q^{2n}e^{2z})(1+q^{2n})^2}$$

The cases where $\alpha = q$ or $\alpha = -q$ are treated similarly with

$$f(u) = \frac{u\Theta(u)}{\Theta(q^{-1/2}u)}$$

and

$$f(u) = \frac{u\Theta(u)}{\Theta(iq^{-1/2}u)}$$

respectively.

STRICT MULTIPLICATIVITY. – The following theorem, also known (in an equivalent form) as the *Witten Conjecture*, was proven first by C. Taubes [12], then by R. Bott and C. Taubes [1].

Theorem 2. *Let P be a principal G-bundle over an oriented manifold B, where G is a compact connected Lie group, and suppose G acts on a compact Spin-manifold M. Then*

$$\varphi(P \times_G M) = \varphi(B)\varphi(M).$$

For the history of the conjecture, cf. [8].

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