Overview

Every hypertoric variety admits a natural embedding into a Lawrence toric variety. We describe the resulting tropicalization explicitly and prove that it is faithful, in that the map from the Berkovich analytification admits a unique continuous section. This yields many new examples of faithful tropicalizations. These examples occur in every even dimension, and include cotangent bundles to products of projective spaces as well as several singular varieties.

Tropicalization

Given a variety \( X \) with an embedding into a toric variety \( Y \), we obtain a tropicalization \( \text{Trop}(X) \). The tropicalization is the support of a balanced finite polyhedral complex, and may be thought of as the image of \( X(K) \) under the coordinate-wise valuation map, where \( K \) is an algebraically closed complete valued field.

More precisely, \( \text{Trop}(X) \) is the continuous image of the Berkovich analytification \( X^\text{an} \) under a tropicalization map to the vector space spanned by the cocharacter lattice of the dense torus in \( Y \). Different embeddings of \( X \) into toric varieties yield distinct tropicalizations. It is a result of Foster-Gross-Payne that \( X^\text{an} \) is the limit of the inverse system of all tropicalizations. It raises the question: How well does a particular tropicalization approximate the topology of \( X^\text{an} \)?

Definition. A tropicalization \( \text{Trop}(X) \) is faithful if the tropicalization map \( X^\text{an} \rightarrow \text{Trop}(X) \) admits a continuous section.

Baker-Payne-Rabinoff first studied faithful tropicalizations for curves. Cueto-Habich-Werner provide the first example in higher dimensions: the Grassmannian of planes \( \text{Gr}(2, n) \) is faithfully tropicalized by its Plücker embedding.

For a given variety \( X \), there is no known procedure for obtaining a faithful tropicalization, not is it known that a faithful tropicalization must even exist. However, recent work of Gubler-Rabinoff-Werner provides explicit combinatorial criteria to verify whether a particular tropicalization of \( X \) is faithful. Application of these criteria requires an explicit description of \( \text{Trop}(X) \) as a polyhedral complex.

Hypertoric varieties

Let \( M \) be a lattice, and let \( T = \text{Spec} \, K[M] \). Let \( A \) be an arrangement of rational hyperplanes in the vector space \( M_R = M \otimes \mathbb{R} \). A choice of coorientation for each hyperplane in \( A \) determines a polyhedron \( P \) and hence a quasiprojective \( T \)-toric variety \( Y_P \).

The arrangement \( A \) also defines a variety \( \mathcal{M}_A \), called a hypertoric variety, which is a “hyperkähler” or “quaternionic” analogue of \( Y_P \). The relationship between \( \mathcal{M}_A \) and \( A \) is analogous to the relationship between \( Y_P \) and \( P \), in that geometric information about \( \mathcal{M}_A \) is encoded in the combinatorics of the arrangement \( A \). While \( \mathcal{M}_A \) is not a toric variety in general, it comes equipped with a natural embedding into a toric variety \( \mathcal{B}_A \), called the Lawrence toric variety.

Faithful tropicalization of \( \mathcal{M}_A \)

We shall use \( \text{Trop}(\mathcal{M}_A) \) to denote the tropicalization of \( \mathcal{M}_A \) given by its embedding in \( \mathcal{B}_A \).

The intersection of the hypertoric variety \( \mathcal{M}_A \) with a torus orbit in \( \mathcal{B}_A \) is a linear subvariety of that torus. Thus, we obtain the following description of \( \text{Trop}(\mathcal{M}_A) \) as a union of Bergman fans.

Theorem (K ’16). The tropicalization \( \text{Trop}(\mathcal{M}_A) \) may be given the structure of a finite polyhedral complex, which is balanced when all cones are given weight one. It has cones \( C_{F, R} \) indexed by a flat \( F \) of \( A_0 \), a face \( R \) of the localization \( A_F \), and a flag of flats \( F \) of the restriction \( A_{F_0} \). These cones satisfy

\[
\dim C_{F, R} = d + \ell(F) - \text{codim } R,
\]

and

\[
C_{F', R'} \subseteq C_{F, R}
\]

if and only if the following conditions hold:

1. \( F \subseteq F' \);
2. \( R' \subseteq R \);
3. \( F' \) is a flat in \( F \), and \( \text{trunc}_F(F') \) is a refinement of \( F' \).

In particular, the maximal cones of \( \text{Trop}(\mathcal{M}_A) \) correspond to maximal flags of flats of \( A \). Since the truncation of a maximal flag is maximal, we conclude that the closure of a maximal cone \( C_F \) is a union of cones, each of which is maximal in its respective stratum. This allows us to apply the combinatorial criteria of Gubler-Rabinoff-Werner to obtain the following.

Theorem (K ’16). There is a unique continuous section \( \text{Trop}(\mathcal{M}_A) \rightarrow \mathcal{M}_A^\text{an} \) of the tropicalization map.