HOMEWORK 1

1) Evans section 2.1 problem 1

2) Solve the following equation in $\mathbb{R}^2$:

$$u_t - xu_x = 0, \qquad u(0,x) = 3x$$

Hint: Try to modify the function $z(s)$ defined in class to

$$z(s) = u(t+s, xe^{-s}).$$

3) i) Show that if $u \in C^1(\mathbb{R}^2)$ is a solution of the transport equation

$$u_t + bu_x = 0 \tag{1}$$

then

$$\int_{\mathbb{R}^2} u(\phi_t + b\phi_x)dxdt = 0 \tag{2}$$

for all $\phi \in C_c^\infty(\mathbb{R}^2)$.

Hint: Pick a ball B so that ϕ is identically 0 outside of B, and integrate by parts in B.

ii) Motivated by i), we say that a function u is a weak solution of (1) if (2) holds for all $\phi \in C_c^\infty(\mathbb{R}^2)$. Show that $u(t,x) = |x - bt|$ is a weak solution of (1), even though it is not differentiable everywhere.

Hint: Divide B into two domains B_1 and B_2 so that $x - bt > 0$ in B_1 and $x - bt < 0$ in B_2. Integrate by parts separately in B_1 and B_2. Carefully note that the terms inside B_1 and B_2 become 0, and that the boundary terms on the line $x - bt = 0$ are also 0.