MATH 533 MIDTERM

Solve at least three of the following questions. If you complete more than three questions, your grade will be based on the best three.

Problem 1: Find a solution $u \in C^2([0,\infty) \times \mathbb{R})$ to the equation

$$u_{tt} - u_{tx} = 0, \quad u(0,x) = 0, u_t(0,x) = x$$

Is the solution unique?

Problem 2: Let $U \subset \mathbb{R}^n$ be a bounded domain, $u : U \to \mathbb{R}$ a harmonic function, and $x_0 \in U$. Find an explicit constant C independent of U, u and x_0 so that

$$|Du(x_0)| \leq \frac{C}{d(x_0, \partial U)} \sup_{x \in U} |u(x)|$$

where $d(x_0, \partial U)$ is the distance from the point x_0 to the boundary of U. Hint: The mean value formula might be helpful.

Problem 3: Let $U \subset \mathbb{R}^n$ be a bounded domain. Consider the Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + u = f & (t,x) \in (0,\infty) \times U \\ u(t,x) = 0 & (t,x) \in [0,\infty) \times \partial U \\ u(0,x) = g(x), \quad u_t(0,x) = h(x) \end{cases} \tag{1}$$

Show that (1) has at most one solution.

Problem 4: Consider the heat equation

$$\begin{cases} u_t - \Delta u = 0 & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = g(x) \end{cases} \tag{2}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded.

Recall that one solution of (2) is given by

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y) dy \tag{3}$$

where Φ is the fundamental solution given by

$$\Phi(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$$

i) Is the solution (3) unique? Briefly justify.

ii) Find an explicit constant C that only depends on g so that the solution (3) satisfies

$$|u(t,x)| \leq C$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$.

iii) Construct an example of a function g to show that in general the solution (3) does not decay in time at any rate, i.e. there is no $\alpha > 0$ and constant C so that

$$|u(t,x)| \leq \frac{C}{t^\alpha}$$