Deterministic Approximation of Stochastic Evolutionary Dynamics

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Abstract— **Deterministic dynamical systems are often used in economic models to approximate stochastic systems with large numbers of agents. A number of papers have provided conditions that guarantee that the deterministic models are in fact good approximations to the stochastic models. Much of this work has concentrated on the continuous time case with systems of differential equations approximating discrete time stochastic systems, although some important early work in this field considered discrete time approximations. A crucial aspect of the existing work is the assumption that the stochastic models involve agents of finitely many types. However, many existing economic models assume agent types may take on infinitely many distinct values. For example, some models assume agents hold divisible amounts of money or goods, and therefore agent types form a continuum. In this paper we examine discrete time deterministic approximations of stochastic systems, and we allow agent attributes to be described by infinitely many types. If the set of types that describe agents is a continuum, then individuals, in some sense, are unique, and it is not obvious that the stochastic models that rely on random matching of agents as the source of uncertainty can be approximated in a deterministic manner. Indeed, we give two examples that show why a law of large numbers may not lead to deterministic approximations. In the positive direction, we provide conditions that allow for good deterministic approximations even in the case of a continuum of types.**

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I. Introduction

Many economic models study systems of interacting agents or players whose strategies or types change over time as a result of interaction of the individual players. Players are assumed to be randomly matched in some fashion during a time period, they interact, perhaps earn some reward, and then change type or strategy. If the population is large, the models that describe the dynamics of the system are often deterministic in spite of the fact that the underlying mechanism that drives the dynamics, the matching, is random.

A fundamental problem is to understand the extent to which the deterministic model approximates the underlying stochastic model. There are a number of papers that study this question in the setting of evolutionary game theory and evolutionary dynamics. In that setting, the deterministic model is often a system of ordinary differential equations. In other cases, discrete time deterministic dynamical systems are used to approximate the stochastic systems. The choice between continuous time approximations and discrete time approximations may depend upon the mechanism that pair agents for interaction. One feature that the discrete time approximation allows is the simultaneous change in type or strategy of many agents. In some cases, the time structure of the stochastic model may not allow subdivision into smaller and smaller periods. For example, in biological models, a species may mate once per year, and therefore discrete time periods are built into the model.

For continuous time deterministic approximation of stochastic systems, the matching mechanism is often fairly simple because only two agents are paired at any given instant, and pairings are assumed to be independent. In fact, evolutionary game dynamics with finite sets of players need not involve random matching at all. For example, in the paper of Benaïm and Weibull [4] one player is selected at random from a finite population and given an opportunity to modify strategy in a manner that depends on the current state of the population. Thus they define a discrete time Markov process with time interval δ and transition probabilities that allow exactly one individual to change strategy during each period. They then show that this stochastic process is approximated by a continuous time process as δ tends to zero. In contrast to the evolutionary mechanism of Benaïm and Weibull, this paper considers games for which random matching plays an essential role.

If one wishes to allow simultaneous pairings of players, the independence of pairings is lost and things become more complicated. It is possible to overcome this difficulty by assuming, as in [8], that a distribution of pairings is determined by matching individuals two at a time with replacement. If this process is carried out N times for a population of size $2N$, one obtains a distribution of pairs that is multinomial. Since this type of matching process generates binomial or multinomial random variables, computation of quantities such as variance, are fairly straightforward.

One crucial aspect of many of these models is the assumption that the number of distinct strategies or types is finite. There are a number of situations however in which the number of strategies or types is not finite. In fact, Aumann [3] notes that the assumption of finite types is often unrealistic as individual agents may be expected to posses distinct "consumption bundles". For games that allow players to select from a continuum of strategies, the set of types for individual players will obviously be infinite. In these games a deterministic model is often assumed at the outset. For example, the paper by J. Oechssler and F. Riedel [16] introduces the concept of evolutionary robustness, an analog of ESS for games with a continuum of strategies. In that paper the deterministic payoff is given by

$$E(P,Q) = \int_S \int_S f(x,y)Q(dy)P(dx), \quad (1)$$

with population P against population Q . Similar examples can be found in [10] and [15]. In some monetary models, [12], [13], [18], that allow divisible money, agents are assumed to be bilaterally matched during discrete time periods, and they exchange amounts of money based on a specified bargaining rule. Here again, the set of types of individuals will be infinite.

The main purpose of this paper is to determine conditions under which a deterministic approximation of a stochastic dynamical system of bilaterally matched agents of infinitely many types can be carried out. In particular, we want to provide conditions that justify a deterministic model such as the one given by equation (1) above as a limit with randomly matched players selecting from a continuum of strategies. As Aumann pointed out in [3], one must pay particular attention to the relationship between the relative size or cardinality of the set of agents and the set of types. In order to obtain valid deterministic models, the number of agents must, in some sense, be large compared with the number of types. Since the number of types is infinite one must take some care to make this precise. A second difficulty that must be overcome involves the information available to agents in deciding on changes in type. If the updating rule for types for individuals is not local in the sense that it may depend upon the interaction of other agents in the population, then deterministic approximation may not be possible.

In the next section we present two examples for which a deterministic approximation to the stochastic model is not appropriate. We then briefly mention a model from monetary theory with agent attributes described by a continuum of types. In this model, the type changing rule for individual agents is local in a sense we make precise, and so our weak law of large numbers (WLLN) can be applied to justify the law of motion for the distribution of types within the population.

II. Examples

In this section we provide three examples. The first example argues that a deterministic approximation of the aggregate behavior of a large population is not reasonable if agents attributes are described by an infinite set of discrete types, and the population has a high degree of heterogeneity. The second example presents a subtle point that is often ignored in applications of the WLLN. In particular the WLLN does not imply point-wise convergence of the underlying probability distribution, and this lack of point-wise convergence can make deterministic models inappropriate approximations to the stochastic models if type changing rules for agents are not local. Finally, we present an example of a model from monetary theory in which, although all agents are unique and agent attributes are described by a continuum of types, a deterministic model is a good approximation to the underlying stochastic model.

Consider a sequence of populations, A_N, consisting of $2N$ agents. Suppose the number of distinct types represented in the population A_N is K_N, and assume that there are at least two agents of each type in the population A_N. If we bilaterally match the agents in the population, we obtain N pairs. The number of possible types of pairs is equal to the number of distinct ordered pairs (i, j) with $i \leq j$, and i, j running from 1 to K_N. In other words, the number of distinct types of pairs equals $K_N(K_N+1)/2$. For any single bilateral match of the agents, the number of realized pairs equals N. Therefore, if $K_N(K_N+1)/2$ grows faster than N, so

$$\frac{K_N(K_N+1)/2}{N} \to \infty \text{ as } N \to \infty,$$

then each match will realize, as the population size tends to infinity, a smaller and smaller fraction of possible pair types.

If, in this model, two distinct pair types are very different in an economic sense, it would make little sense to approximate the stochastic matching process with a deterministic process. For example, suppose we describe agent type by listing 5 out of 50 distinct traits. This results in approximately 2.1×10^6 distinct types. The number of types of pairs is therefore approximately 2.2×10^{12}. Trying to approximate the fraction of pair types in a random match of agents in a population this heterogeneous with a deterministic model would make little sense even if the population size were equal to the human population of the earth.

On the other hand, if two distinct pair types could be considered close in some sense, then it may be possible to approximate the underlying stochastic model with a deterministic model. For example, if the type of an agent is defined by the amount of money held in \$1 units, then an agent holding \$100 could be considered "close" to an agent holding \$101. In this case, it may well be possible to use a deterministic model to approximate a random matching model.

There are other situations in which a deterministic model will not provide a good approximation to the underlying stochastic model even if we consider finite time horizons, arbitrarily large populations, and large numbers of agents representing each type. The weak law of large numbers is a statement about convergence in probability, and this convergence may not be strong enough to allow a deterministic approximation.

Evolutionary models specify a type or strategy changing process that determines how agents change type from one period to the next. These type changing rules depend upon the type of each agent in a matched pair. However, the type changing rule may also depend upon how other agents in the population interact. For example, a frequent assumption in learning in games models is a type changing rule that depends upon the aggregate outcome of the game for a period. Examples can be found in the paper by M. Kandori, G. Mailath and R. Rob [11], and the paper of A. Robson and F. Vega-Redondo [17]. Specifically, the type changing rule might depend upon the average payoff to all individuals playing a particular strategy. The following example uses a type changing rule of this nature to show why it may be impossible to eliminate large random fluctuations in population characteristics from one period to the next by taking the population size arbitrarily large.

Consider a population of players of two types, A and B who play a matching game. The players have the option to select $Continue$, or to select $Stop$. Once they select $Stop$, they are out of the game. The set of all players who select $Continue$ are simultaneously bilaterally randomly matched. If agents in a matched pair are of like type they both receive a payoff of 1, and if they are of different types they both receive a payoff of -1. Suppose that after a match of all players, a player updates her strategy to *Stop* if the aggregate (total) payoff for all players is negative *and* her individual payoff is negative. Otherwise, she selects $Continue$ and participates in the next round. In particular, all players select $Continue$ if the aggregate payoff is equal to 0. We let X_{AB} denote the fraction of pairs of type $\{A, B\}$ in a bilateral match. The realized value of X_{AB} will of course depend upon the randomly selected match, and it is easy to explicitly write down the probability distribution for X_{AB}. However for the purpose of this example, we simply note that if the population size is large and there are equal numbers of A and B types in the population, the probability that X_{AB} is greater than $1/2$ is approximately equal to $1/2$.

Suppose we start play with an equal number of A and B types, and all players initially select $Continue$. Players are bilaterally matched for the period. With probability approximately $1/2$, the value of X_{AB} will be greater than $1/2$ and hence the average payoff for all players will be negative. Hence an equal number of A and B players will select *Stop*, and drop out. The remaining players will $Continue$. On the other hand, with probability approximately $1/2$ the value of X_{AB} will be less than or equal to $1/2$, the realized average payoff will be positive, and all players will $Continue$. In this case the state of the system for the start of the second period will be identical to the state for the first period. It should be clear that the size of the continuing population will go to zero exponentially fast.

Now suppose we try to approximate this stochastic system with a deterministic system. As the population size tends to infinity, with equal numbers of A and B types, the expected fraction of $\{A, B\}$ pairs in a random match of the entire population tends to $1/2$. In fact the random variable X_{AB} tends to the constant $1/2$ in distribution by the law of large numbers established by Boylan in [5]. If we assume a deterministic model such that $X_{AB} = 1/2$ for *each bilateral match of the entire population* then the average payoff for all players will always *exactly* equal 0, and no player will ever change strategy. Hence, even over a single period, the stochastic system is very different than the deterministic system obtained by applying the law of large numbers.

A brief moment of reflection should make it clear what is going on here. In the above model, as population size tends to infinity, the limit average payoff for all players will be 0, but the variance is *never* equal to 0. Hence with probability approximately $1/2$ the players will observe a negative average payoff and "learn" that the game is not a good deal. The "bad deal" is reinforced for those players who receive a negative payoff, and they decide to quit. One might argue that players should not use the observed payoff for a single period, but rather use the expected average payoff where *expected* means with respect to all possible matches. This argument seems somehow at odds with the idea that players will learn from observation rather than deduce, through mathematical computation, a reasonable strategy.

It is also useful at this point to give a mathematical explanation of the problem this example presents for the use of the law of large numbers. Let $F_{AB}(x)$ denote the distribution function of the random variable X_{AB}. Let $F(x)$ denote the distribution function of the constant random variable[1], equal to $1/2$ so that

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1/2 \\ 1 & \text{if} \quad x \geq 1/2 \end{cases} .$$

Although the random variable X_{AB} tends to the constant $1/2$ in distribution, the distribution function $F_{AB}(x)$ does not tend to the distribution function $F(x)$ point-wise. In particular, we do not have convergence at $x = 1/2$.

The above example contains two messages. First, it is possible to construct very simple models such that the law of large numbers will not be enough to justify approximating a stochastic model by a deterministic model. Second, suppose one is trying to model a population of players who are sensitive to an observed aggregate outcome that they perceive in simplistic "good-bad" terms. If the dividing line between "good" and "bad" is equal to the mean of a matching process, then variance takes over and deterministic models may not apply.

Finally, we present a model that has been frequently used in monetary theory with a continuum of types present in the population. Instead of repeating verbatim an existing model, we write down a general model that captures the essential feature found in a number of papers, [12], [13], [18], that use matching as the source of randomness.

Consider agents with attributes described by a pair (τ_i, m_i) where the τ_i take values in a finite set and the m_i take values in $[0, \infty)$. The values τ_i might indicate the type of good bought or sold by an agent, and we shall refer to τ_i as the *good* type. The value m_i might indicate the amount of money held by an agent, and we shall refer to m_i as the $money\ holdings$ of the agent. We suppose that during discrete time periods agents in the population are bilaterally randomly matched with every possible match for a finite population equally likely. As a result of an interaction the *good* type does not change. However the $money\ holdings$

[1]Suppose one assumes an infinite population of equal proportions of type A and type B players. Further, assume this infinite population is somehow randomly matched so that exactly $1/2$ of the resulting pairs is of type $\{A, B\}$. Then the distribution function for the random variable X_{AB} will be the function $F(x)$. Matching schemes for a continuum of agents that satisfy this proportionality condition have been developed by Alós-Ferrer [1], [2] and Duffie and Sun [9].

change according to a rule

$$\widetilde{m_i} = m_i + \delta(\tau_i, \tau_j) d(m_i, m_j),$$

where $\widetilde{m_i}$ denotes the next period holdings of a type τ_i agent with *money holdings* m_i who is matched with a type τ_j agent with *money holdings* m_j. The function $\delta(\cdot, \cdot)$ assumes values of 1, -1, or 0 depending upon the values of the arguments. If $\delta(\tau_i, \tau_j) = 1(-1)$ we think of the i agent as a seller (buyer). Let A_N denote a sequence of populations of agents with types described as above. We denote the distribution of types of the finite population A_N by $(\rho_{N,i}, F_{N,i}(x))$ where $\rho_{N,i}$ denotes the fraction of agents of *good* type τ_i in the population, and $F_{N,i}(x)$ denotes the distribution of *money holdings* of the *good* type τ_i agents. Suppose the distributions $(\rho_{N,i}, F_{N,i}(x))$ converge pointwise to a distribution $(\rho_i, F_i(x))$. We show in this paper that, with mild restrictions on the function $d(\cdot, \cdot)$ that describes how much money changes hands, it is possible to write down a simple deterministic law of motion for the distribution of agents in the population that is a good approximation to the stochastic system defined through the bilateral random matching of the finite population for sufficiently large N.

III. Deterministic Equations of Motion

Before we present our approximation results, we describe the deterministic laws of motion that we would like to obtain as approximations. The approximation theorems require certain assumptions on the manner in which population size increases with respect to type, and it will be useful to see one reason for the assumption in the simpler deterministic setting. We first describe the case of discrete types and then the case of continuous types. In the discrete case, the results hold for finitely and infinitely many types. Finally, we describe the equation of motion in the case of mixed types.

A. Discrete Types

Denote the set of discrete types of agents by $K = \{1, 2, 3, ..., k, ...\}$. Let τ denote the function that indicates how types change when a type i agent is matched with a type j agent. So $\tau(i, j) = k$ means that if a type i agent is matched with a type j agent then the type i agent becomes a type k agent for the next time period.

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), ..., \alpha_k(t), ...)$ be a vector of real valued functions of t that satisfy

$$0 \leq \alpha_i(t) \leq 1 \text{ and } \sum_{i=1}^{\infty} \alpha_i(t) = 1. \qquad (2)$$

The $\alpha_i(t)$ represent the fraction of the population of type i at time t.

We now relate the functions, $\alpha_i(t)$, at different times to obtain a deterministic dynamical system. Let $\alpha(0)$ be a vector that satisfies equations (2). Let

$$\Gamma_i = \tau^{-1}(i) = \{(j, k) \,|\tau(j, k) = i\}$$

Recursively define

$$\alpha_i(t + 1) = \sum_{(j,k) \in \Gamma_i} \alpha_j(t) \alpha_k(t). \qquad (3)$$

First note that there is no problem with convergence because the α_i are positive and sum to 1. It may not be at all obvious that the recursive equation above, (3), is consistent with the conditions on the $\alpha_i(t)$, but a short computation will show that the recursive equation makes sense.

Notice that our definition of the dynamical system is completely deterministic - there is no randomness. On the other hand, precisely because the vector $\alpha(t)$ satisfies conditions (2), we can give the system a probabilistic interpretation.

The optimal policy and equilibrium problems commonly used in economic models generally start with a deterministic dynamical system of the type we have described *along with* the probabilistic interpretation above. The associated optimal policy problem, in our notation, is to select an interaction policy, τ, to optimize some discounted reward function defined in terms of the vector $\alpha(t)$ (and perhaps some additional random process).

B. Continuous Types

We next consider the case of a continuum of types. We suppose the type of an agent is described by a real number x. The analog of the α_i which sum to 1 is a probability distribution which we denote by $F(x)$ and which we assume is continuous. Hence $F(x)$ indicates the fraction of the population with type less than or equal to x. Let τ denote the type changing rule so that $\tau(r, s) = x$ means that a type r agent paired with a type s agent becomes a type x agent. We do not assume that τ is symmetric.

Let $F(t, x)$ be a continuous probability distribution (in x) and let $P(t)$ denote the probability measure associated with $F(t, x)$. Define a deterministic dynamical system with state space equal to the set of continuous distributions by

$$F(t + 1, x) = P(t) \times P(t) \left(\tau^{-1}((-\infty, x])\right).$$

We assume the type changing rule, τ, is continuous. The continuity assumption immediately implies that $F(t + 1, x)$ is continuous and increasing in x. To show that the above equation defines a probability distribution, we proceed as in the discrete case. We must show

$$\lim_{x \to \infty} F(t + 1, x) = 1.$$

This follows since

$$\lim_{x \to \infty} \tau^{-1}((-\infty, x]) = \mathbb{R} \times \mathbb{R}$$

and $F(t, x)$ is a probability distribution.

C. Mixed Types

Finally, we consider the case of agent attributes described by a pair of variables, one discrete and one continuous. For simplicity, we shall assume the type changing rule for the discrete type variable depends only on the discrete variable values, but that the type changing rule for the continuous variable may depend upon both the discrete variable and the continuous variable. This is the situation most often encountered in existing models, and the third example in Section II satisfies this property. In particular

$$\tau\left((j, u), (k, v)\right) = (\ell, x)$$

indicates that a discrete type j agent with continuous type u who is matched with a discrete type k agent with continuous type v , changes to a discrete type ℓ agent with continuous type x. The value of ℓ depends only upon j and k , but the value of x depends upon $j, k, u,$ and v. We indicate the dependence by $x = \tau_{jk}(u, v)$, and we assume τ_{jk} is continuous for all j, k. We also write, by a slight abuse of notation, $\tau(j, k) = \ell$ to indicate that a discrete type j agent matched with a discrete type k agent becomes a discrete type ℓ agent. Let

$$\Gamma_{(\ell,x)} = \{((j,u),(k,v))\,|\tau(j,k) = \ell \text{ and } \tau_{jk}(u,v) \le x\}\,.$$

Let the fraction of agents with discrete type j and continuous type less than or equal to x at time t be $F(t,(j,x))$ with corresponding probability measure written $P(t)$. We can now state the deterministic equation of motion.

$$F\left(t+1,(\ell,x)\right) = P(t) \times P(t)\left(\Gamma_{(\ell,x)}\right).$$

By combining the arguments presented above for discrete and continuous types, it is straightforward to show that this equation does define a distribution function for the type ℓ agents at time $t+1$. Since the measure $P(t)$ can be written as a sum of the measures P_j , P restricted to $\{j\} \times \mathbb{R}$, the above expression for $F(t+1,(\ell,x))$ can be written in the form

$$F\left(t+1,(\ell,x)\right) = \sum_{\tau(j,k)=\ell} P_j(t) \times P_k(t)\left(\Upsilon_{jk}(x)\right)$$

where

$$\Upsilon_{jk}(x) = \{((j,u),(k,v))\,|\tau_{jk}(u,v) \le x\}\,.$$

IV. Stochastic Dynamical System for Discrete Types

In this section we consider the case of populations of agents whose relevant attributes are described by a countable set of types. Our goal is to determine conditions so that stochastic dynamic systems of interacting agents in these populations can be approximated by a deterministic system.

We prove a weak law of large numbers for random matching with a countable (finite or infinite) number of types of agents. As anticipated in the paper [3] by Aumann, we must pay close attention to the manner in which the number of types and the number of agents jointly tend to infinity. As we add types, the dimension of the multivariate distribution of number (or fraction) of pairs of a particular type increases. From [3],

> "Thus as we add traders, the space under consideration changes, and in such a context it is not clear what "approach" means. It was because of this difficulty that Debreu and Scarf postulated a fixed finite number of types of traders; together with some other assumptions, this enabled them to work with a fixed finite dimensional space."

In fact it is not too difficult to deal with the complication of infinitely many types, and it is crucial that one be able to do this for applications in random matching models.

Consider a population, A_N, of $2N$ agents. Each agent assumes a type for a period, and we can therefore denote agents by a pair (i, ℓ_i) where the ℓ_i take values in the infinite set $\mathbb{N}$ of non-negative integers. Instead of using the awkward notation (i, ℓ_i) we simply indicate an agent by ℓ_i with the understanding that ℓ_i and ℓ_j are to be considered distinct if $i \ne j$ in counting arguments even if the values of ℓ_i and ℓ_j are identical. Thus, we indicate our set of agents by

$$A_N = \{\ell_1, \ell_2, \ell_3, ..., \ell_{2N}\}\,.$$

Let

$$\alpha_N(\ell) = \frac{1}{2N}\text{card}\,\{i\,|\ell_i = \ell\}$$

denote the fraction of agents of type ℓ in the population A_N. We assume there exist positive numbers $\alpha(\ell)$ that sum to 1 such that

$$\alpha_N(\ell) \to \alpha(\ell)$$

uniformly in ℓ as $N \to \infty$.

By the Glivenko-Cantelli theorem, given the numbers $\alpha(\ell)$ that sum to 1, the population defined by selecting $2N$ random variables from the distribution defined by the $\alpha(\ell)$ will have the desired property.

Let Φ denote the set of bilateral matches of the population of $2N$ agents. We make Φ into a probability space by placing the discrete uniform distribution on this set. Any bilateral match of $2N$ agents can be defined by a permutation of the elements of the set $\{1, 2, 3, ..., 2N\}$ as a sequence of N pairs of matched agents,

$$\left(\ell_{\pi(1)}, \ell_{\pi(2)}\right)\left(\ell_{\pi(3)}, \ell_{\pi(4)}\right)\cdots\left(\ell_{\pi(2N-1)}, \ell_{\pi(2N)}\right).$$

Two distinct permutations may of course define the same bilateral match. Hence, one associates a match with an equivalence class of permutations, although we shall just write π to refer to the match as well as one of the permutations that defines it. The set Φ therefore has $((2N)!)/(N!2^N)$ elements.

For a given match π, let $X_{N,\pi}(\ell, m)$ denote the fraction of pairs in a given random match of type $\{\ell, m\}$. The random variables $X_{N,\cdot}(\ell, m)$ are of course dependent. We want to obtain a weak law of large numbers for these random variables. The basic idea is to compute the mean and variance of the variables and then apply Chebychev's inequality. Let

$$\mu_N(\ell, m) = E_\pi\left[X_{N,\cdot}(\ell, m)\right]$$

and

$$var_N(\ell, m) = E_\pi\left[\left(X_{N,\cdot}(\ell, m)\right)^2\right] - \left(\mu_N(\ell, m)\right)^2.$$

Write I_B to denote the indicator function of the set B. Then we can write

$$X_{N,\pi}(\ell, m) = \frac{1}{N}\sum_{k=1}^{N} I_{\{\ell,m\}}\left\{\ell_{\pi(2k-1)}, \ell_{\pi(2k)}\right\}.$$

This expression allows us to easily compute the expected value, since we need only calculate

$$E_\pi\left[I_{\{\ell,m\}}\left\{\ell_{\pi(2k-1)}, \ell_{\pi(2k)}\right\}\right]$$

and sum. This expectation is computed by elementary combinatorial arguments and Poisson's Theorem on the probability of drawing a given sequence of multicolored balls from an urn (see for example [7]). We obtain

$$\mu_N(\ell,m) = \begin{cases} 2\alpha_N(\ell)\alpha_N(m)\frac{2N}{2N-1} & \text{if} \quad \ell \neq m \\ (\alpha_N(\ell))^2 - \left(\frac{\rho_N(\ell)}{2N-1}\right) & \text{if} \quad \ell = m \end{cases} \tag{4}$$

where $\rho_N(\ell) = \alpha_N(\ell)\left(\alpha_N(\ell) - 1\right)$.

The computation of the variance is a bit more complicated and because of limitations of space is omitted. We simply state the result: $var_N(\ell,m) \sim \mathcal{O}\left(\frac{1}{N}\right)$.

We can now state and prove the WLLN for random matching with a countable infinity of types.

Theorem 1: Weak Law of Large Numbers.

Let $\alpha(\ell)$ be a sequence of positive real numbers with

$$\sum_{\ell=1}^{\infty}\alpha(\ell) = 1.$$

Suppose A_N is a sequence of finite populations of agents such that each agent is assigned a type $\ell \in \mathbb{N}$, and $\alpha_N(\ell)$ denotes the fraction of agents of type ℓ in A_N. Assume $\alpha_N(\ell) \to \alpha(\ell)$ uniformly in ℓ, and put

$$\mu(\ell,m) = \begin{cases} 2\alpha(\ell)\alpha(m) & \text{if} \quad \ell \neq m \\ (\alpha(\ell))^2 & \text{if} \quad \ell = m \end{cases}.$$

If $X_{N,\pi}(\ell,m)$ denotes the random variable equal to the number of pairs of type $\{\ell,m\}$ in a random match of the agents in A_N, then $X_{N,\cdot}(\ell,m)$ converges in probability to $\mu(\ell,m)$ uniformly in ℓ and m. In other words for every $\epsilon > 0$,

$$\lim_{N\to\infty}\sup_{\ell,m}\text{Prob}\left(|X_{N,\cdot}(\ell,m) - \mu(\ell,m)| > \epsilon\right) = 0$$

Proof: Given $\epsilon > 0$ we estimate $\text{Prob}\left(\{|X_N(\ell,m) - \mu(\ell,m)| > \varepsilon\}\right)$ using Chebyshev's inequality and the triangle inequality. We obtain

$$\begin{aligned}&\text{Prob}\left(\{|X_N(\ell,m) - \mu(\ell,m)| > \varepsilon\}\right) \\ &\leq \frac{var_N(\ell,m)}{\left(\epsilon - |\mu_N(\ell,m) - \mu(\ell,m)|\right)^2}\end{aligned}$$

for N sufficiently large. Using the equation (4) for the mean we see that $\mu_N(\ell,m)$ converges uniformly in ℓ and m to $\mu(\ell,m)$. Since the variance converges to 0 uniformly in ℓ and m as N tends to ∞, the proof is complete. ■

We now discuss discrete time stochastic processes defined by random matching and a type or strategy changing law. Suppose $\tau : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defines a rule for type change. For simplicity, we assume τ does not depend upon time. Various rules have been proposed for changes in type or strategy. One possibility is a mimic rule as in Corradi and Sarin [8], so that one agent might assume the type of her partner. Another possibility is a learning rule that allows agents to change type based on a payoff of the interaction between the agents. Similar to learning rules, are bargaining rules such as those used in monetary models (see for example [12], [13], [18]).

Given a finite population, A_N, bilateral random matching together with the type changing rule defines a stochastic process. Let $\alpha_N(t,\ell)$ denote the fraction of agents of type ℓ at time t and let π denote a random bilateral match. Then

$$\begin{aligned}&\alpha_N(t+1,\ell) = \\ &\sum_{\tau(p,p)=\ell} X_{N,\pi}(p,p) + \\ &\frac{1}{2}\left(\sum_{\substack{\tau(p,q)=\ell \\ p\neq q}} X_{N,\pi}(p,q) + \sum_{\substack{\tau(q,p)=\ell \\ p\neq q}} X_{N,\pi}(p,q)\right)\end{aligned} \tag{5}$$

The left side of the equation is a random variable on the space of matches. Note that the right side of the equation depends on the values of $\alpha_N(t,\ell)$ since the values of the random variables $X_{N,\pi}(p,q)$ depend upon $\alpha_N(t,\ell)$. Given an initial distribution of types $\alpha_N(0,\ell)$ in the population A_N, this expression for $\alpha_N(t,\ell)$ completely describes the process where we assume the random matching at time t is independent of matching at time $\widetilde{t}$ for $t \neq \widetilde{t}$. Recall that we regard Φ as the probability space for the random matching with the discrete uniform distribution. Hence, over a finite time horizon, $t = 0,1,2,3,...,T$, we regard the stochastic process $\alpha_N(t,\ell)$ with given initial distribution $\alpha_N(0,\ell)$ as a random function on the product space Φ^T. The WLLN for $X_{N,\cdot}(\ell,m)$ gives the following result.

Theorem 2: Weak Law of Large Numbers for Random Matching Process

Let τ be a type changing rule on $\mathbb{N}\times\mathbb{N}$. Put $\Gamma_\ell = \{(p,q)\,|\tau(p,q) = \ell\}$. Let $\alpha(t,\ell)$ be the deterministic dynamical system defined by the initial state $\alpha(0,\ell)$ and the transition law

$$\alpha(t+1,\ell) = \sum_{(p,q)\in\Gamma_\ell}\alpha(t,p)\alpha(t,q)$$

The stochastic process $\alpha_N(t,\ell)$ defined by equation (5) above converges in distribution to the deterministic process on any finite time interval $\mathcal{T}$.

V. Stochastic Dynamical Systems for Continuous Types

We now consider stochastic systems of interacting agents with the type of an agent defined by a continuous set of real values. During each discrete time period, the entire population of agents is assumed to be bilaterally matched. Each interaction of two agents results in a modification of the types of the interacting agents for the start of the next period. We shall assume a deterministic rule for the change in type of agents that depends only upon the current period types of the interacting agents. The assumption of a deterministic type changing rule can be relaxed to allow for stochastic type changing rules.

The interaction of agents may also result in a reward or payoff for the agents. In describing equilibria of a sequential decision process, dynamic programming methods are typically employed and the Bellman equation that defines equilibrium can be justified by the law of large numbers applied to the stochastic rewards for interacting agents.

To keep the exposition and notation as simple as possible, we assume in this section that player types are defined by a single real variable. We shall indicate in the next section how this restriction can be lifted. We shall also use a limited amount of terminology and notation from the theory of random measures although we do not assume familiarity with that subject.

It is very important to keep in mind that there are several probability spaces involved in the following discussion. When we speak of a distribution of types of agents, denoted by $F(x)$ or $F_N(x)$, there is implicitly a probability space associated with F and F_N, but these probability spaces are not usually mentioned explicitly. The probability space associated with the random matching is usually explicitly mentioned and it is denoted by Φ. Hence when we speak of convergence below, one must pay particular attention to the space on which the convergence occurs. We generally follow the convention of not indicating the probability space variable when talking about random variables. However, because we shall be discussing *random distributions,* we shall often indicate the matching space variable explicitly.

Suppose now A_N is a population of $2N$ agents or players, and that to each player is associated a type which is a real number. This real number may represent a strategy selected by the player as in a game with continuous strategy sets, or the real number may represent some attribute such as capital held by the player. An agent should be indicated by a pair, (i, x_i), rather than simply a value x_i because it will be important to be able to distinguish two agents who are assigned the same value. However, by a slight abuse of notation we shall simply use x_i to denote an agent rather than the somewhat awkward notation (i, x_i). Thus, if $i \neq j$, x_i will always be assumed to be distinct from x_j. It will be important to keep this notational convenience in mind for the counting arguments, and we shall use it without further mention. We use the term "*value* of x_i" to explicitly refer to the real number associated with an agent in case of possible confusion. We therefore write our set of distinct agents as

$$A_N = \{x_1, x_2, x_3, ..., x_{2N}\},$$

although the *values* of the x_i are not necessarily distinct. The sequence of sets, A_N, defines distributions of the player types, and we write

$$F_N(x) = \frac{1}{2N} \text{card}\{i \,|x_i \leq x\}.$$

We shall assume that there is a limiting distribution, $F(x)$ so that $F_N(x) \to F(x)$ uniformly in x.

Given a probability distribution $F(x)$ one can always construct populations A_N having the desired property by letting A_N be the set of realizations of $2N$ independent random variables with distribution F. By the Glivenko-Cantelli theorem, the empirical distribution, $F_N(x)$, converges to $F(x)$ uniformly in x.

Our basic goal is to understand the extent to which stochastic evolutionary processes that pair players or agents through a random bilateral matching process can be approximated by a deterministic process. The stochastic processes generally involve a reward or payoff function and a type changing function. Here we concentrate on the type changing function, but the reward function can be analyzed in exactly the same manner.

Consider a finite population A_N of $2N$ players with each player assigned a real number x_i as above. Suppose that when player x_i is paired with player x_j, player x_i changes type to $\tau(x_i, x_j)$ where the function $\tau(\cdot,\cdot)$ depends only on the *values* of x_i and x_j. For a single bilateral match of all players there will be $2N$ values (not necessarily distinct) assumed by the type changing function - two values for each pair in the realized match since τ is not assumed symmetric in its arguments. A match can be identified with a set of pairs,

$$\left\{(x_{\pi(1)}, x_{\pi(2)}), (x_{\pi(3)}, x_{\pi(4)}), \cdots, (x_{\pi(2n-1)}, x_{\pi(2N)})\right\},$$

where π is a permutation of $\{1, 2, 3, ..., 2N\}$. Note that two distinct permutations may define the same bilateral match. Thus we identify a bilateral match with the equivalence class of all permutations that define the same match, and we indicate a match simply by one of the permutations that defines it. The cardinality of the set of bilateral matches of our set of $2N$ agents is $[(2N)!]/N!2^N$. Let $G_{N,\pi}(x)$ be the distribution of the new types for a match π. If we let

$$\widetilde{\tau}(\pi, i) = \begin{cases} \tau(x_{\pi(i)}, x_{\pi(i+1)}) & \text{if} \quad i \text{ is odd} \\ \tau(x_{\pi(i)}, x_{\pi(i-1)}) & \text{if} \quad i \text{ is even} \end{cases},$$

then

$$G_{N,\pi}(x) = \frac{\text{card}\{i \,|\widetilde{\tau}(\pi, i) \leq x\}}{2N}.$$

In other words, $G_{N,\pi}$ denotes the distribution of player types in the population after a match. The definition makes sense since this value is independent of the permutation used to define a match. We have used the subscript π to emphasize that the distribution of the payoff will depend explicitly on the match, and so here we are indicating the probability space variable. Hence, $G_{N,\pi}(x)$ is a random distribution defined on the probability space, Φ, of all bilateral matches of the $2N$ players with the uniform discrete distribution on Φ. This random distribution depends explicitly on the type changing function, τ , and the original distribution of values of A_N.

Our first goal is to show that as the population size tends to infinity, the random distribution, $G_{N,\pi}(x)$, tends to a constant (non-random) distribution that can be written in terms of the measure associated with the limiting distribution of $F_N(x)$. To this end we derive the following weak law of large numbers (WLLN) for the distribution of new types.

Theorem 3: Weak Law of Large Numbers: Continuous Types.

Let A_N be a sequence of populations such that the empirical distribution of types, $F_N(x)$, converges uniformly in x to a continuous distribution $F(x)$ with corresponding probability measure P. Suppose $\tau(x, y)$ is a type changing function that specifies the new type of an individual of type x when paired with a individual of type y. Assume τ is continuous. Then

$$G_{N,\pi}(x) \Rightarrow P \times P\left(\tau^{-1}(-\infty, x]\right) \quad (6)$$

where the convergence is in probability and uniform in x. The theorem says that for a fixed x value, the real valued random variable $G_{N,\cdot}(x)$ converges in probability to the constant on the right side of this equation. The probability variable of the random variable $G_{N,\cdot}(x)$ is π.

Proof: We outline the basic idea of the proof which follows from Chebychev's inequality and a computation of the mean and variance of the random variable $G_{N,\cdot}(x)$ on the probability space of bilateral matches. Note that for fixed x, $G_{N,\pi}(x)$, can be regarded as a real valued random variable, and if we regard x as a variable, $G_{N,\pi}(x)$ can be regarded as a random distribution. Let I_A be the indicator function of the set A. First note that we can rewrite the empirical distribution of new types as

$$G_{N,\pi}(x) = \frac{1}{2N} \sum_{i=1}^{2N} \left[I_{(-\infty,x]}\left(\widetilde{\tau}(\pi, i)\right)\right]. \quad (7)$$

Thus the computation of the expected value and variance of $G_{N,\cdot}(x)$ can be reduced to a computation of the expected value of the indicator functions and their squares in the above expression. We shall often identify distributions and the measures they define to help the exposition flow more smoothly.

Note that the sum of indicator functions,

$$I_{(-\infty,x]}\left(\widetilde{\tau}(\pi, i)\right),$$

can be regarded as a random distribution on Φ. Although the random variables in the sum in equation (7) are dependent, they have identical distribution. Calculating the expected value of the indicator function and summing gives,

$$E_\pi\left[G_{N,\cdot}(x)\right] = \frac{\text{card}\left\{(j,k)\,|j \neq k, \tau(x_j, x_k) \leq x\right\}}{2N(2N-1)}.$$

We want to compute the limiting value of the distribution above as N tends to infinity. First note that as N tends to infinity we have asymptotically

$$E_\pi\left[G_{N,\cdot}(x)\right] \sim \frac{\text{card}\left\{(j,k)\,|\tau(x_j, x_k) \leq x\right\}}{(2N)^2}.$$

Let X_N, Y_N denote independent random variables with distribution $F_N(x)$ and X, Y denote independent random variables with distribution $F(x)$. Now

$$\frac{\text{card}\left\{(j,k)\,|\tau(x_j, x_k) \leq x\right\}}{(2N)^2}$$

equals the probability that two random draws, X_N and Y_N (with replacement) from the population A_N will satisfy $\tau(X_N, Y_N) \leq x$. Since we assume the type changing function, τ, is continuous, the Continuous Mapping Theorem implies

$$\tau(X_N, Y_N) \Rightarrow \tau(X, Y)$$

where the convergence is in distribution. Since the distribution $F(x)$ is assumed continuous, the distribution function of the random variable $\tau(X, Y)$ is continuous. It therefore follows that the distribution function of $\tau(X_N, Y_N)$ converges uniformly in x to the distribution function of $\tau(X, Y)$. In other words,

$$\frac{\text{card}\left\{(i,j)\,|\tau(x_i, x_j) \leq x\right\}}{(2N)^2} \Rightarrow (P \times P)\left(\tau(X,Y) \leq x\right)$$

uniformly in x, and where we have written P for the probability measure associated with the distribution F. Since there are $2N$ terms in the sum in equation (7), it follows that

$$E_\pi\left[G_{N,\cdot}(x)\right] \rightarrow (P \times P)\left(\tau(X,Y) \leq x\right)$$

uniformly in x.

We also need an estimate for the variance of $G_{N,\cdot}(x)$, but since this computation is a bit more involved we omit the details due to space constraints. The computation shows that $var\left(G_{N,\cdot}(x)\right)$ converges to 0 like $1/N$ uniformly in x.

By Chebychev's inequality, exactly as in the discrete case, this gives convergence in probability

$$G_{N,\cdot}(x) \Rightarrow (P \times P)\left(\{(X,Y)\,|\tau(X,Y) \leq x\}\right)$$

uniformly in x. ∎

We next discuss the stochastic process defined by the random matching of a finite set of agents whose types are defined by a continuous parameter and give a weak law of large numbers that justifies the approximation by a deterministic process over a finite (discrete) time horizon. The continuous type case is very similar to the discrete type case and so we leave most details to the reader. We first recall the deterministic dynamical system that serves as our approximation. We shall assume types are given by non-negative real numbers and τ is a continuous function that defines the change of types for interacting agents. Let $F(0, x)$ be an initial continuous distribution of types. Put

$$F(t+1, x) = P(t) \times P(t)\left(\tau^{-1}(-\infty, x]\right) \quad (8)$$

where $P(t)$ denotes the probability measure associated with the distribution function $F(t, x)$.

Next we define the stochastic process with state $F_N(t, x)$ generated by the random matching and an initial population A_N with initial empirical distribution $F_N(0, x)$. During each discrete time period the agents are bilaterally randomly matched and types are updated by the type changing rule τ and the realization of a match. Hence $F_N(t, x)$ denotes the empirical distribution of the types of the agents at time t. We assume matching during different time periods is independent. Let $\mathcal{T}$ denote the finite time interval $t = 0, 1, 2, ..., T$. We want to approximate $F_N(t, x)$ over the interval $\mathcal{T}$. Just as in the discrete case we obtain a weak law of large numbers for the stochastic process.

Theorem 4: Weak Law of Large Numbers: Random Matching Process with Continuous Types

Let A_N be a sequence of finite populations such that the distribution of types $F_N(x)$ converges uniformly to the continuous distribution $F(x)$, and let $\tau(x, y)$ be a continuous type changing rule. Let $F(t,x)$ be the deterministic dynamical system defined by

$$F(t+1,x) = P(t) \times P(t)\left(\tau^{-1}(-\infty,x]\right)$$

with initial distribution $F(x)$. Then the stochastic system, $F_N(t,x)$, converges in probability to $F(t,x)$ uniformly in x and uniformly on $\mathcal{T}$.

The above analysis focused on the process that describes the state of the evolutionary game. However, the same analysis can be carried out to obtain deterministic approximations to the aggregate payoff for the game as well. This will lead to an expression that is essentially identical to equation (1). Suppose the payoff to an agent of type x when matched with an agent of type y is given by $f(x,y)$, and suppose $f(\cdot,\cdot)$ is continuous. One then obtains a weak law of large numbers for the distribution of the payoffs exactly as above. Hence the deterministic payoffs commonly assumed in games with continuous strategy spaces can be regarded as limiting cases of randomly matched players selecting from a continuum of strategies.

It is important to be able to compute the payoff to agents in a deterministic manner for many sequential decision problems since one can then write down a deterministic Bellman equation to determine equilibrium. Examples can be found in papers of Molico [13] and Zhu [18]. In these papers, agents are characterized by infinitely many types, and during each discrete time period the agents are assumed to be bilaterally matched. The models assume a continuum of agents to justify the deterministic Bellman equation as well as the deterministic law of motion. Both the type changing rules and the payoffs to individual agents are continuous functions. In both of these papers the type of an agent is in fact defined by a discrete parameter and a continuous parameter. In the next section we shall see how this additional complication can be dealt with by using the results we have obtained thus far.

VI. Mixed Types

In this section we consider the case of populations of agents with attributes described by a pair of types (ℓ,x) with ℓ discrete and x continuous. As in the previous section, one should denote an individual agent as (i,ℓ_i,x_i), but again we simply indicate an agent by (ℓ_i,x_i). We denote the set of agents by $A_N = \{(\ell_1,x_1),(\ell_2,x_2),(\ell_3,x_3),...,(\ell_{2N},x_{2N})\}$ with the usual understanding that (ℓ_i,x_i) and (ℓ_j,x_j) are distinct if $i \neq j$ even if the values are identical. Let

$$F_N((\ell,x)) = \text{card}\left\{i\,|(\ell_i,x_i)\in A_N\ ,\ x_i \le x \text{ and } \ell_i = \ell\right\}.$$

We use the notation $F_N((\ell,x))$ with the double parenthesis to avoid confusion with the notation of the previous section where the first variable denoted time. The type of an agent is now given by the pair (ℓ,x) and hence we think of (ℓ,x) as the type variable. In the context of games with continuous strategy spaces one may think of the index ℓ as indicating the population type and x indicating the strategy of a player in the population type ℓ. We shall define a stochastic process for repeated matching and use the notation $F(t,(\ell,x))$ to denote the distribution of type (ℓ,x) agents at time t.

As in the previous sections, we assume the empirical distributions converge to a distribution function so,

$$\lim_{N\to\infty} F_N((\ell,x)) = F((\ell,x)) \tag{9}$$

uniformly in ℓ and x. We assume $F((\ell,x))$ is continuous is x. We assume that τ is a type changing function that satisfies the conditions of Section III-C. Hence

$$\tau\left((j,u),(k,v)\right) = (\ell,x)$$

where the value ℓ depends only on j and k, but the value x depends upon $j,k,u,$ and v. We also use the notation

$$\Gamma_{(\ell,x)} =$$

$$\left\{((j,u),(k,v))\,|\tau\left((j,u),(k,v)\right) = (\ell,y) \text{ with } y \le x\right\}$$

Let P denote the probability measure associated with the distribution function $F((\ell,x))$.

Now suppose we are given a set of agents A_N with the distribution of agents described by $F_N((\ell,x))$. Let π denote a random match of the agents in A_N, and let $G_{N,\pi}((\ell,x))$ denote the new (after the match π is made) distribution of agents. We wish to find the limit of the distributions $G_{N,\pi}((\ell,x))$ as N tends to infinity under the assumption (9). We have the following weak law of large numbers for matching with mixed types.

Theorem 5: WLLN: Mixed Types

Let A_N be a population of $2N$ agents of mixed type (ℓ,x). Assume the distributions of types, $F_N((\ell,x))$ satisfy condition (9) given above, and P is the measure associated with the limit distribution function $F((\ell,x))$. Assume $F((\ell,x))$ is continuous in x and τ is continuous in the continuous type variable. Then

$$G_{N,\cdot}((\ell,x)) \to P \times P\left(\Gamma_{(\ell,x)}\right) \tag{10}$$

as N tends to infinity. The convergence is in probability and uniform in ℓ and x.

Recall that $G_{N,\cdot}((\ell,x))$ should be regarded as a random distribution defined on the probability space of random bilateral matches of the elements of A_N with each match equally likely. Hence the result says that this distribution converges in probability to the constant (non-random) distribution function on the right side of the expression (10).

Proof: The proof follows by combining the arguments used in the proofs of the WLLN for discrete types and continuous types. Since no new arguments are required, we simply indicate the initial steps in the proof. Introduce the notation

$$\delta^{jk}_{\ell m} = \begin{cases} 1 & \text{if } j=\ell \text{ and } k=m \\ 0 & \text{otherwise} \end{cases},$$

and as in the previous section,

$$\widetilde{\tau}(\pi,i) = \begin{cases} \delta^{jk}_{\ell_{\pi(i)}\ell_{\pi(i+1)}} \cdot (\tau(j,k),\tau_{jk}(x_j,x_k)) & i \text{ odd} \\ \delta^{jk}_{\ell_{\pi(i)}\ell_{\pi(i-1)}} \cdot (\tau(j,k),\tau_{jk}(x_j,x_k)) & i \text{ even} \end{cases}.$$

Then we can write the random distribution, $G_{N,\cdot}$, as

$$G_{N,\pi}((\ell,x)) = \frac{1}{2N}\sum_{i=1}^{2N}\left[I_{(\ell,(-\infty,x])}\left(\widetilde{\tau}(\pi,i)\right)\right].$$

Now proceed as in the previous section to compute the mean and variance, and use Chebychev's inequality to obtain the result as before. ∎

VII. Conclusion

The main contributions of the paper are laws of large numbers for random matching processes of a population with agents of infinitely many types. Models of this type are ubiquitous in the economics literature, and yet when agents are characterized by an infinite set of types, very specific assumptions must be made for the application of the law of large numbers to make sense.

The first example of the paper shows that the set of distinct types of agents must grow slowly relative to the growth of the set of agents itself if one is to have any hope of using a deterministic approximation to the stochastic process that describes interacting agents when random matching determines the uncertainty in the model. On the other hand, if type is defined by a continuous parameter and the type changing rule obeys some continuity properties, then it may be possible to approximate the stochastic systems by deterministic systems as our results show.

The second example shows that the qualitative nature of the type changing rule is a crucial factor in deterministic approximation. A type changing rule is *local* if agents change type in a manner that depends only upon the types of two interacting agents. Our results assume local type changing rules, and our second example shows that chaotic stochastic behavior can occur if type changing rules are not local. It may not be obvious if a particular type changing rule is strictly local in some cases. For example, suppose agents from a population of size $2N$ are repeatedly bilaterally matched, and after each match they receive a payoff that depends upon bargaining between the agents. The agents then change type for the next period in a way that depends upon this bargaining. Assume the objective of each player is to maximize expected payoff over a fixed time horizon. If the bargaining rule depends upon expected future payoffs, the type changing rule may not be local in the sense we describe above. Obtaining deterministic limit dynamical systems appears to be quite difficult in these cases.

An important goal for future research will be to obtain a better understanding of the qualitative nature of systems with large numbers of interacting agents when the nature of interaction of matched individuals has both a local and global component.

In particular, we would like to find general conditions on the type changing rule that would allow a limiting deterministic approximation to the underlying stochastic system.

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