Fixed Point Theorems in Topology and Geometry

A Senior Thesis Submitted to the Department of Mathematics In Partial Fulfillment of the Requirements for the Departmental Honors Baccalaureate

By Morgan Schreffler

Millersville, Pennsylvania
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This Senior Thesis was completed in the Department of Mathematics, defended before and approved by the following members of the Thesis committee

A. Ronald Umble, Ph.D (Thesis Advisor)
Professor of Mathematics,
Millersville University

B. Bruce Ikenaga, Ph.D
Associate Professor of Mathematics,
Millersville University

C. Zhigang Han, Ph.D
Assistant Professor of Mathematics,
Millersville University
Abstract

Assuming nothing about the mathematical background of the reader, selected fixed-point theorems in topology and transformational geometry are rigorously developed and discussed.
# Fixed Point Theorems in Topology and Geometry

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1 Introduction

This paper is an exposition of the Brouwer Fixed-Point Theorem of topology and the Three Points Theorem of transformational plane geometry. If we consider a set $X$ and a function $f : X \rightarrow X$, a fixed point of $f$ is a point $x \in X$ such that $f(x) = x$. Brouwer’s Fixed-Point Theorem states that every continuous function from the $n$-ball $B^n$ to itself has at least one fixed point. An isometry is a bijective function from $\mathbb{R}^2$ to itself which preserves distance. Although the Three Points Theorem is not itself a fixed-point theorem, it is a direct consequence of the following fixed-point theorem: An isometry with three non-collinear fixed points is the identity. The Three Points Theorem states that if two isometries agree at three non-collinear points, they are equal.

We shall assume the reader is familiar with the most basic elements of set theory, as well as the fundamentals of mathematical proof. Nothing else about the reader’s mathematical background is taken for granted. As a result, Brouwer’s Fixed-Point Theorem and the Three Points Theorem are given a complete and rigorous treatment which will be neither overwhelming nor alienating to any undergraduate math student.

Since the one-dimensional case of the Brouwer Fixed-Point Theorem is the most accessible and intuitive, we shall discuss it first. The development of this theorem is based heavily on topology, so our first task is to develop an understanding of what topology is and how it works. First, topological spaces are sets together with a collection of subsets, called open sets, that satisfy certain axioms. Next, we define connected spaces and continuous functions. Following this, we introduce the idea of topological equivalence, known as homeomorphism. Though it is not entirely relevant until Section 3.5, it is natural to define and explore homeomorphism alongside functions and continuity. Afterwards, the relationship between continuous functions and connectedness is considered, which allows us to prove the Intermediate Value Theorem (IVT). Finally, we prove the one-dimensional case of the Fixed-Point Theorem as a direct consequence of the IVT.

The two-dimensional case of the Brouwer Fixed-Point Theorem and its proof are far less intuitive than their one-dimensional counterparts. First, we need to explore Cartesian products in detail, as well as the topologies they produce, since $\mathbb{R}^2$ is in fact the Cartesian
product of $\mathbb{R}$ with itself. Next we define homotopy, which can be understood as a weak form of “topological equivalence” for continuous functions. Intuitively, two continuous functions are homotopic if one can “melt” into the other without sacrificing continuity during the process. After proving that homotopy is an equivalence relation, we address circle functions - continuous functions which map the circle to itself - and we partition them into equivalence classes based on how many times they “wrap around the circle.” The equivalence class to which a circle function belongs is called its degree, and proving that the degree functions $c_n$ are representatives of the equivalence classes of circle functions is a very difficult problem. This problem is addressed in Appendix A. The degree of circle functions is used to prove the Two-Dimensional No-Retraction Theorem, which states that no retraction from the disk to its boundary exists. The Two-Dimensional No-Retraction Theorem is then used to prove the Two-Dimensional Brouwer Fixed-Point Theorem. Finally, we show that the fixed-point property is, in fact, a topological property, i.e., two homeomorphic spaces either both have or both lack the fixed-point property.

In Section 3, the Two-Dimensional Brouwer Fixed-Point Theorem is shown to be a consequence of the No-Retraction Theorem in two dimensions. In fact, the proof of the Fixed-Point Theorem used in Section 3.5 remains exactly the same, regardless of dimension. The proof of the No-Retraction Theorem in $n$ dimensions also changes little from that used in Section 3.4. What changes are the homotopy classes of continuous functions on the $n$-sphere, as well as the techniques used to obtain them. These homotopy classes are briefly discussed in Section 4.1. Afterward, the No-Retraction Theorem is rewritten to accommodate the $n$-sphere homotopy classes.

As we move into Section 5, our first task is to define a transformation. We then look at a specific type of transformation, called an isometry, which preserves distance. Next, we show that if an isometry fixes two points, then it necessarily fixes the entire line defined by those two points, and if a third point not on that line is also fixed, then the entire plane is fixed! The Three Points Theorem is an immediate consequence.

We finish the main body of the paper by showing that an isometry is the composition of two reflections in distinct intersecting lines if and only if it is a rotation about the intersection point of the two lines. To do this, we must first define a reflection and a rotation and...
prove that they are isometries. Then we show that the aforementioned isometries agree at three non-collinear points, at which point the Three Points Theorem does the rest.

As mentioned earlier, we return to topology in Appendix A for a lengthy and rigorous treatment of Theorem 85. It is treated as an appendix because a large portion of the material therein is used solely to prove Theorem 85. First we introduce the concept of compactness and prove that a set in \( \mathbb{R} \) is compact if and only if it is closed and bounded. We also prove that continuous functions map compact sets to compact sets. This, in turn, allows us to prove the Extreme Value Theorem, as well as a special case of the Lebesgue Number Lemma. Next, we introduce a special type of function called a lifting and, using the Lebesgue Number Lemma, we show that every circle function and every circle homotopy has a lifting. With these theorems we can prove the existence and uniqueness of every circle function’s degree, which is exactly Theorem 85.
2 One-Dimensional Case of the Brouwer Fixed-Point Theorem

To visualize the One-Dimensional Brouwer Fixed-Point Theorem, imagine a unit square situated in the first quadrant of the Cartesian plane with one of its vertices at the origin. Now, suppose we draw the diagonal connecting the bottom-left and upper-right vertices of our square. This diagonal line segment is contained in the line $y = x$. The Brouwer Fixed-Point Theorem tells us that we cannot draw a curve from the left side of the square to the right side of the square without either crossing this diagonal or picking up our pencil (see Figure 1). Thinking of our curve as the graph of a continuous function, every point at which the curve crosses the diagonal $y = x$ is a fixed point.

![Figure 1: Every continuous function $f : [0, 1] \rightarrow [0, 1]$ intersects the line $y = x$.](image_url)
2.1 Topological Spaces and Open Sets

Definition 1 Let $X$ be a set, and let $\mathcal{T}$ be a collection of subsets, called open sets of $X$. Then $\mathcal{T}$ is called a topology on $X$ if the following axioms hold:

A1. $X \in \mathcal{T}$
A2. $\emptyset \in \mathcal{T}$
A3. The intersection of any finite number of open sets is open
A4. The union of any collection of open sets is open.

We call the set $X$ together with a topology $\mathcal{T}$ on $X$ a topological space. When a result does not depend on a particular topology, a topological space $(X, \mathcal{T})$ will usually be denoted $X$ for simplicity.

Example 2 Let $X = \{a, b, c\}$. Consider the two collections of subsets on $X$ as seen in Figure 2. Both collections include $X$ and $\emptyset$, as well as every intersection of subsets. However, Collection 2 includes $\{a\}$ and $\{c\}$, but does not include their union $\{a, c\}$, whereas every union of subsets in Collection 1 is contained in Collection 1. Thus, Collection 1 is an example of a topology on $X$, while Collection 2 is not.

![Figure 2: Two collections of subsets on $\{a, b, c\}$](image)

Definition 3 Let $X$ be a set, and $\mathcal{B}$ a collection of subsets of $X$. We say $\mathcal{B}$ is a basis for a topology on $X$, and each $B_n$ in $\mathcal{B}$ is called a basis element if $\mathcal{B}$ satisfies the following axioms:

B1. For every $x \in X$, there is a set $B \in \mathcal{B}$ such that $x \in B$, and
B2. If $B_1$ and $B_2$ are in $\mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3$ in $\mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
Before we make our first claim, we will need a brief lemma dealing with unions.

**Lemma 4** (*The Union Lemma*) Let \( X \) be a set and \( \mathcal{C} \) be a collection of subsets of \( X \) such that for each \( x \in X \) there is a set \( C_x \) in \( \mathcal{C} \) such that \( x \in C_x \). Then \( \bigcup_{x \in X} C_x = X \).

**Proof.** To prove the sets are equal, we will prove that each is a subset of the other. First, since each \( C_x \) is a subset of \( X \), \( \bigcup_{x \in X} C_x \subseteq X \). Now, suppose \( y \in X \). This implies that there is a subset \( C_y \) in \( \mathcal{C} \) such that \( y \in C_y \subseteq \bigcup_{x \in X} C_x \). Thus, \( X \subseteq \bigcup_{x \in X} C_x \), which completes the proof. ■

**Definition 5** Let \( X \) be a topological space and let \( x \in X \). A **neighborhood** of \( x \) is an open set \( N \) such that \( x \in N \).

**Theorem 6** Let \( X \) be a topological space and let \( A \subseteq X \). The set \( A \) is open in \( X \) if and only if for every \( x \in A \) there is a neighborhood \( U \) of \( x \) such that \( U \subseteq A \).

**Proof.** \((\rightarrow)\) Let \( A \) be open in \( X \). Clearly, for each \( x \in A \), \( A \) is a neighborhood of \( x \). Thus, letting \( U = A \), \( x \in U \subseteq A \) for all \( x \in A \).

\((\leftarrow)\) Let \( A \subseteq X \) such that for each \( x \in A \), there is a neighborhood \( U_x \) such that \( x \in U_x \subseteq A \). Now, by **Lemma 4**, \( A = \bigcup_{x \in A} U_x \). It follows that \( A \) is a union of open sets and is therefore open by **Definition 1:A4**. ■

In **Definition 3**, we called \( \mathcal{B} \) a basis for a topology on \( X \). This is because taking arbitrary unions of elements of \( \mathcal{B} \) allows us to very easily generate a topology over \( X \) in much the same way that bases of vector spaces allow us to easily generate coordinate systems for those spaces. But how do we know that this process actually generates a topology?

**Proposition 7** Let \( X \) be a set, and \( \mathcal{B} \) a basis over \( X \). Let \( T \) be a collection of subsets of \( X \) which includes \( \emptyset \) and which is closed with respect to arbitrary unions of basis elements. Then \( T \) is a topology over \( X \).

**Proof.** We must show that our claim satisfies each axiom of **Definition 1**.

A1. Since every \( x \in X \) is an element of some basis element \( B_x \in \mathcal{B} \), \( X \) is the union of all the basis elements and so \( X \) is in \( T \).

A2. \( \emptyset \) is an element of \( T \) by hypothesis.
A3. We must show that a finite intersection of sets of \( T \) is in \( T \). Let \( U_1, U_2, \ldots U_n \) be either elements of \( T \), \( \emptyset \), or a union of basis elements. Now, consider \( V = U_1 \cap U_2 \cap \ldots \cap U_n \). If any \( U_i = \emptyset \), then \( V = \emptyset \in T \). Thus, we will consider the case in which each \( U_i \) is a union of basis elements.

Let \( x \in V \). Then \( x \in U_i \) for all \( i \) by the definition of intersection. Now, since each \( U_i \) is a union of basis elements, there exists a basis element \( B_i \) such that \( x \in B_i \subseteq U_i \) for each \( i \). Thus, \( x \in B_1 \cap B_2 \cap \ldots \cap B_n \). By using Definition 3: B2 inductively, there must exist a basis element \( B_x \) such that \( x \in B_x \subseteq B_1 \cap B_2 \cap \ldots \cap B_n \subseteq V \). It follows from Lemma 4 that \( V = \bigcup_{x \in V} B_x \). Therefore, \( V \), a finite intersection of sets in \( T \), is a union of basis elements and is consequently a set in \( T \).

A4. To show that the union of any arbitrary collection of sets in \( T \) is in \( T \), consider \( V = \bigcup U_\alpha \), where each \( U_\alpha \) is either \( \emptyset \) or a union of basis elements. If \( U_\alpha = \emptyset \) for each \( \alpha \), then \( V = \emptyset \), and thus is a set in \( T \) by hypothesis. Thus, suppose at least one \( U_\alpha \) is nonempty. It follows that \( V \), which is a union of \( U_\alpha \)'s, is in fact a union of basis elements, proving that an arbitrary union of sets in \( T \) is in \( T \).

Since \( T \) satisfies all four axioms, \( T \) is a topology over \( X \). ■

Now that we have shown that \( \mathcal{B} \) does in fact generate a topology over \( X \), we can formally define the topology generated by \( \mathcal{B} \).

**Definition 8** Let \( X \) be a set and \( \mathcal{B} \) a basis over \( X \). The topology \( \mathcal{T} \) generated by \( \mathcal{B} \) is created by defining the open sets of \( \mathcal{T} \) to be \( \emptyset \) and all sets that are a union of basis elements.

With Definition 8, we can build many interesting topologies over the real numbers. For our purposes, however, we will be chiefly concerned with one particular topology over \( \mathbb{R} \).

**Definition 9** Let \( \mathcal{B} = \{(a, b) \subseteq \mathbb{R} \mid a < b \} \). Every point \( x \in \mathbb{R} \) is also an element of at least one open interval, for example, \((x - 1, x + 1)\). Also, the intersection of two open intervals in \( \mathbb{R} \) is either \( \emptyset \) or another open interval. Thus, \( \mathcal{B} \) satisfies the requirements of a basis. Further, the topology generated by \( \mathcal{B} \) is called the standard topology on \( \mathbb{R} \).
Note: Later in the paper we will be dealing with results over the real numbers. Though the results may hold in other topologies of \( \mathbb{R} \), it will be assumed that we are working with the standard topology over \( \mathbb{R} \).

**Axiom 10** Let \( A \) be a subset of \( \mathbb{R} \). If \( A \) is bounded above, then it has a least upper bound. Further, if \( A \) is bounded below, then it has a greatest lower bound.

### 2.2 Closed Sets; Interior, Exterior, and Boundary

**Definition 11** Let \( X \) be a topological space and \( A \subseteq X \). Then the complement of \( A \) in \( X \), denoted \( A^c \), is the set of all points \( x \in X \) which are not in \( A \). The set \( A^c \) may also be denoted \( X - A \).

**Definition 12** Let \( X \) be a topological space. A subset \( A \) of \( X \) is closed if \( A^c \) is open.

**Theorem 13** Let \((X, T)\) be a topological space. Then the following properties hold:

C1. \( X \) is closed.

C2. \( \emptyset \) is closed.

C3. The intersection of any collection of closed sets is closed.

C4. The union of any finite number of closed sets is closed.

**Proof.**

C1. Since \( \emptyset \) is open by **Definition 1:A2**, \( \emptyset^c = X \) is closed by **Definition 12**.

C2. Since \( X \) is open by **Definition 1:A1**, \( X^c = \emptyset \) is closed.

C3. By **Definition 1:A4**, \( \bigcup T_r \) is open for arbitrary open sets \( T_r \) in \( X \). Thus, \( (\bigcup T_r)^c = \bigcap (T_r)^c \) is closed. Because each \( (T_r)^c \) is closed, it follows that any arbitrary intersection of closed sets is closed.

C4. By **Definition 1:A3**, \( \bigcap_{i=1}^{n} T_i \) is open for arbitrary open sets \( T_i \) in \( X \). Consequently, \( (\bigcap_{i=1}^{n} T_i)^c = \bigcup_{i=1}^{n} (T_i)^c \) is closed. Because each \( (T_i)^c \) is closed, it follows that the union of a finite number of closed sets is closed. ■

**Definition 14** Let \( X \) be a topological space and \( A \subseteq X \). A point \( a \) is an interior point of \( A \) if there exists a neighborhood of \( a \) which contains only points of \( A \). The set of all interior points of \( A \) is called the interior of \( A \) and is denoted \( \text{Int}(A) \).
Theorem 15 A set $A$ is open if and only if all of its points are interior points.

**Proof.** ($\rightarrow$) Let $A$ be an open set. Clearly, $A$ is a neighborhood of each of its points which contains only points of $A$. Thus, all points in $A$ are interior points.

($\leftarrow$) Let $A$ be a set such that all of its points are interior points. Thus, for each $a \in A$, there exists a neighborhood $N_a$ which contains only points in $A$. It follows from Theorem 6 that $A$ is open. ■

Definition 16 A point $y$ is an exterior point of a set $X$ if there exists a neighborhood of $y$ which contains only points of $X^C$. The set of all exterior points of $X$ is called the exterior of $X$ and is denoted $\text{Ext}(X)$.

Definition 17 A point $z$ is a boundary point of a set $X$ if every neighborhood of $z$ contains a point in $X$ and a point in $X^C$. The set of all boundary points of $X$ is called the boundary of $X$ and is denoted $\partial(X)$.

Lemma 18 Let $X$ be a topological space and $A \subseteq X$. Then $\partial(A) = \partial(A^C)$.

**Proof.** We will show that each is a subset of the other. Let $d \in \partial(A)$. Then every neighborhood $N(d)$ of $d$ contains a point in $A = (A^C)^C$ and a point in $A^C$, which means $d \in \partial(A^C)$. The proof of $\partial(A^C) \subseteq \partial(A)$ is similar. ■

Definition 19 Let $X$ and $Y$ be sets and let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of subsets of $X$. If $X \cap Y = \emptyset$, then we say $X$ and $Y$ are disjoint. Further, if $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha, \beta \in \mathcal{A}$ such that $\alpha \neq \beta$, then we say that the sets in the collection are mutually disjoint.

Definition 20 Let $X$ be a set and let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of subsets of $X$. If $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is mutually disjoint and $\bigcup_{\alpha \in \mathcal{A}}X_\alpha = X$, then $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is called a partition of $X$.

Theorem 21 Let $X$ be a topological space and $A \subseteq X$. Then $\text{Int}(A)$, $\partial(A)$, and $\text{Ext}(A)$ form a partition of $X$ (see Figure 3).

**Proof.** Let $X$ be a topological space, $A \subseteq X$, and $x \in X$. If $x \in \text{Int}(A)$, then there exists a neighborhood $N(x)$ which contains only points in $A$, so $x \in A$. Thus, no neighborhood of $x$ contains only points in $A^C$, and the neighborhood $N(x)$ contains no points in $A^C$, so $x \notin \text{Ext}(A)$ and $x \notin \partial(A)$. Similarly, $x \in \text{Ext}(A)$ implies $x \notin \text{Int}(A)$ and $x \notin \partial(A)$. 

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Figure 3: $\text{Int}(A)$, $\partial(A)$, and $\text{Ext}(A)$ form a partition of $X$.

Now, suppose $x \in \partial(A)$. Then every neighborhood of $x$ contains a point in $A$ and a point in $A^C$. Thus, no neighborhood contains only points in $A$, and no neighborhood exists which contains only points in $A^C$. Hence, $x \notin \text{Int}(A)$ and $x \notin \text{Ext}(A)$.

Finally, suppose $x \notin \text{Int}(A)$, $x \notin \text{Ext}(A)$, and $x \notin \partial(A)$. Then $x \notin \text{Int}(A)$ tells us that every neighborhood of $x$ contains a point in $A^C$, and $x \notin \text{Ext}(A)$ tells us that every neighborhood of $x$ contains a point in $A$. Since every neighborhood of $x$ contains both a point in $A$ and a point in $A^C$, it follows that $x \in \partial(A)$, which contradicts our assumption.

Because we have shown that $\text{Int}(A)$, $\text{Ext}(A)$, and $\partial(A)$ are mutually disjoint, and that their union is $X$, they form a partition of $X$. ■

Theorem 22 A set $A$ is closed if and only if it contains all of its boundary points.

Proof. (←) Let $A$ be a closed set. Suppose there is a point $d \in \partial(A)$ such that $d \notin A$. Then $d$ is an element of $A^C$, which is open by Definition 12. Now, $A^C$ is a neighborhood of $d$ which contains no point in $A$. Thus, $d \notin \partial(A)$. This contradicts our assumption that $d \in \partial(A)$, so $A$ contains all of its boundary points.

(←) Let $A$ be a set which contains all of its boundary points. Then it is clear that $A^C$ contains none of its boundary points. Thus, $A^C$ is made up entirely of interior points, which makes $A^C$ open by Theorem 15. It follows from Definition 12 that $A$ is closed. ■
2.3 Subspace Topology

Proposition 23 Let \((X, \mathcal{T})\) be a topological space and \(Y\) a subset of \(X\). Then \(T_Y = \{U \cap Y \mid U \in \mathcal{T}\}\) is a topology on \(Y\).

Proof. As in Proposition 7, we must verify that each axiom of Definition 1 is satisfied.

A1. \(X \in \mathcal{T}\) and \(Y = X \cap Y\), so \(Y\) is a set in \(T_Y\).
A2. \(\emptyset \in \mathcal{T}\) and \(\emptyset = \emptyset \cap Y\), so \(\emptyset\) is a set in \(T_Y\).
A3. To show that a finite intersection of sets in \(T_Y\) is in \(T_Y\), consider sets \(T_1, T_2, \ldots, T_n\) in \(T_Y\). Then, for each \(i\) there exists \(U_i \in \mathcal{T}\) such that \(T_i = U_i \cap Y\) by hypothesis. Now,

\[
T_1 \cap T_2 \cap \ldots \cap T_n = (U_1 \cap Y) \cap (U_2 \cap Y) \cap \ldots \cap (U_n \cap Y) \\
= (U_1 \cap U_2 \cap \ldots \cap U_n) \cap Y. \tag{1}
\]

Since \(U_i \in \mathcal{T}\) for each \(i\), \(\bigcap_{i=1}^n U_i\) is also an element of \(\mathcal{T}\). This, combined with (1), proves that \(\bigcap_{i=1}^n T_i\) is a set in \(T_Y\).
A4. To show that an arbitrary union of sets in \(T_Y\) is in \(T_Y\), consider a collection of sets \(\{T_\alpha\}\), all of which are in \(T_Y\). By hypothesis, for each \(\alpha\) there exists a set \(U_\alpha \in \mathcal{T}\) such that \(T_\alpha = U_\alpha \cap Y\). Hence,

\[
\bigcup T_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y. \tag{2}
\]

Because each \(U_\alpha\) is in \(\mathcal{T}\), \(\bigcup U_\alpha \in \mathcal{T}\). This, combined with (2), proves that \(\bigcup T_\alpha\) is a set in \(T_Y\).

Since all four axioms are satisfied, \(T_Y\) is a topology over \(X\). ■

Definition 24 Let \((X, \mathcal{T})\) be a topological space and \(Y\) a subset of \(X\). The subspace topology on \(Y\) is defined to be \(T_Y = \{U \cap Y \mid U \in \mathcal{T}\}\), and \(Y\) is called a subspace of \(X\) (see Figure 4).

Theorem 25 Let \(X\) be a topological space and \(\mathcal{B}\) be a basis for the topology on \(X\). If \(Y \subseteq X\), then the collection \(\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}\) is a basis for the subspace topology on \(Y\).

Proof. We will prove that \(\mathcal{B}_Y\) satisfies the axioms of a basis.

B1. Since every \(x \in X\) is in some basis element in \(\mathcal{B}\), it follows that every \(y \in Y\) is in the intersection of \(Y\) and a basis element of \(\mathcal{B}\). Thus, every \(y \in Y\) is in a set \(B \in \mathcal{B}_Y\).
Figure 4: For a set $U$ open in $X$, $U \cap Y$ is open in the subspace topology on $Y$.

B2. Let $B_1, B_2 \in \mathcal{B}_Y$ and let $y \in B_1 \cap B_2$. Then $y \in (B_a \cap Y) \cap (B_b \cap Y)$ for some $B_a, B_b \in \mathcal{B}$. Consequently, $y \in (B_a \cap B_b)$. Since $\mathcal{B}$ is a basis, there exists a $B_c \in \mathcal{B}$ such that $y \in B_c \subseteq (B_a \cap B_b)$. Let $B_3 = B_c \cap Y$. Hence,

$$y \in B_3 = B_c \cap Y \subseteq (B_a \cap B_b) \cap Y = (B_a \cap Y) \cap (B_b \cap Y) = B_1 \cap B_2,$$

and the proof is complete. ■

Note: When $Y \subseteq X$, we denote the complement of a set $C$ in the subspace topology of $Y$ by $Y - C$, and for the complement of $C$ in the “superspace” topology on $X$ we use $C^c$.

**Theorem 26** Let $X$ be a topological space and let $Y \subset X$ have the subspace topology. $C \subseteq Y$ is closed in $Y$ if and only if $C = D \cap Y$ for some closed set $D \subseteq X$.

**Proof.** ($\rightarrow$) Let $C$ be a closed subset of $Y$. Then, by definition, $Y - C$ is open in $Y$. The definition of the subspace topology (Definition 24) tells us that $Y - C = D \cap Y$ for some open set $D \subseteq X$. Now,
\[ C = Y - (Y - C) \]
\[ = Y - (D \cap Y) \]
\[ = (Y - Y) \cup (Y - D) \]
\[ = \emptyset \cup (Y - D) \]
\[ = Y - D \]
\[ = Y \cap D^c. \]

Since \( D \) is open in \( X \), \( D^c \) is closed in \( X \).

\((\leftarrow)\) Let \( C = D \cap Y \) for some closed set \( D \subseteq X \). Then,
\[ Y - C = Y - (D \cap Y) \]
\[ = (Y - D) \cup (Y - Y) \]
\[ = (Y - D) \cup \emptyset \]
\[ = Y - D \]
\[ = Y \cap D^c. \]

Because \( D \) is closed in \( X \), it follows that \( D^c \) is open in \( X \), making \( Y - C = Y \cap D^c \) open in \( Y \). Hence, \( C \) is closed in the subspace topology on \( Y \).

\[ \blacksquare \]

2.4 Connectedness

**Definition 27** A topological space \( X \) is said to be **disconnected** if there exists a pair of disjoint, nonempty open subsets whose union is \( X \).

**Definition 28** A subset \( A \) of a topological space \( X \) is **disconnected in** \( X \) if \( A \) is disconnected in the subspace topology on \( A \).

**Definition 29** A topological space \( X \) is **connected** if it is not disconnected. Further, a subset \( A \) of \( X \) is considered **connected in** \( X \) if \( A \) is not disconnected in the subspace topology.

**Theorem 30** Let \( X \) be a topological space. \( X \) is connected if and only if \( \emptyset \) and \( X \) are the only subsets of \( X \) that are both open and closed.
Proof. \((\rightarrow)\) Let \(X\) be connected. Suppose there exists a subset \(Y\) of \(X\) such that \(Y \neq \emptyset\), \(Y \neq X\), and \(Y\) is both open and closed in \(X\). Since \(Y\) is closed, by definition \(Y^c\) is open. Clearly, \(Y \cap Y^c = \emptyset\) and \(Y \cup Y^c = X\). Hence, \(Y\) and \(Y^c\) are disjoint, nonempty sets whose union is \(X\), so \(X\) is disconnected, which contradicts our hypothesis.

\((\leftarrow)\) Let \(X\) be a topological space in which the only subsets both open and closed are \(\emptyset\) and \(X\). Now, suppose \(X\) is disconnected. Then, by definition there exist a pair of disjoint, nonempty open subsets \(A\) and \(B\) such that \(A \cup B = X\). Since \(A\) and \(B\) are disjoint, \(A^c = B\), which is open. Thus \(A\) is closed, making \(A\) both open and closed. And, since \(A \neq \emptyset\) and \(B \neq \emptyset\), it follows that \(A \neq X\). But, the existence of \(A\) contradicts our hypothesis, so \(X\) is connected. \(\blacksquare\)

Lemma 31 Let \(A\) be a subset of a topological space \(X\). Then \(A\) is disconnected in \(X\) if and only if there exist open sets \(U\) and \(V\) in \(X\) such that the following conditions hold:

1. \(A \subseteq U \cup V\),
2. \(U \cap V \subseteq A^c\),
3. \(U \cap A \neq \emptyset\),
4. \(V \cap A \neq \emptyset\)

Proof. \((\rightarrow)\) Let \(A\) be disconnected in \(X\). Then, by Definition 27, there exists a pair of nonempty, disjoint open sets \(B\) and \(C\) in \(A\) such that \(B \cup C = A\). Now, since \(B\) and \(C\) are open in \(A\), there exist by Definition 24 open sets \(U\) and \(V\) in \(X\) such that \((3) U \cap A = B \neq \emptyset\) and \((4) V \cap A = C \neq \emptyset\). Hence,

\[
a \in A \implies a \in (B \cup C)
\]

\[
\implies a \in [(U \cap A) \cup (V \cap A)]
\]

\[
\implies [a \in (U \cap A)] \lor [a \in (V \cap A)]
\]

\[
\implies [(a \in U) \land (a \in A)] \lor [(a \in V) \land (a \in A)]
\]

\[
\implies (a \in U) \lor (a \in V)
\]

\[
\implies a \in (U \cup V),
\]
showing that (1) $A \subseteq U \cup V$. Next, we will prove (2) $U \cap V \subseteq A^c$ by appealing to the contrapositive. Since $B$ and $C$ are disjoint, $B \cap C = \emptyset$, which implies $A$ is not a subset of $B \cap C$. Thus,

$$a \in A \implies a \notin (B \cap C)$$

$$\implies \neg (a \in B \cap C)$$

$$\implies \neg (a \in B \land a \in C)$$

$$\implies \neg [(a \in U \cap A) \land (a \in V \cap A)]$$

$$\implies \neg [(a \in U \land a \in A) \land (a \in V \land a \in A)]$$

$$\implies \neg (a \in U \cap V)$$

$$\implies a \notin U \cap V.$$

This tells us (2) $U \cap V \subseteq A^c$. Thus, if $A$ is disconnected in $X$, there exist open sets $U$ and $V$ which satisfy the requirements in the lemma.

$(\leftarrow)$ Let $A$, $U$, and $V$ be open sets in $X$ such that (1) $A \subseteq U \cup V$, (2) $U \cap V \subseteq A^c$, (3) $U \cap A \neq \emptyset$, and (4) $V \cap A \neq \emptyset$. Let $B = U \cap A$ and $C = V \cap A$. Then,

$$B \cap C = (U \cap A) \cap (V \cap A)$$

$$= U \cap V \cap A$$

$$\subseteq A^c \cap A$$

$$= \emptyset.$$

Now, to show that $A = B \cup C$, we will prove that each is a subset of the other.

$$a \in A \implies a \in U \cup V$$

$$\implies (a \in U) \lor (a \in V)$$

$$\implies (a \in U \land a \in A) \lor (a \in V \land a \in A)$$

$$\implies (a \in B) \lor (a \in C)$$

$$\implies a \in B \cup C,$$
and,

\[
a \in B \cup C \implies a \in [(U \cap A) \cup (V \cap A)]\]
\[
\implies (a \in U \cap A) \lor (a \in V \cap A)\]
\[
\implies (a \in U \land a \in A) \lor (a \in V \land a \in A)\]
\[
\implies (a \in A) \lor (a \in A)\]
\[
\implies a \in A.
\]

Thus, \(B\) and \(C\) are disjoint open subsets whose union is \(A\), so that \(A\) is disconnected by Definition 27.

**Definition 32** Let \(U,V\) be open in \(X\). If \(U\) and \(V\) satisfy the four properties in Lemma 31, we say \(U\) and \(V\) form a separation of \(A\) in \(X\) (see Figure 5).

![Figure 5: The sets \(U\) and \(V\) form a separation of \(A\) in \(X\).](image)

**Lemma 33** Let \(A\) be a subset of a topological space \(X\). \(A\) is disconnected if and only if there exist closed sets \(F\) and \(G\) in \(X\) that satisfy the same properties as \(U\) and \(V\) in Lemma 31.
2.5 Functions and Continuity

Definition 34 For any sets $A$ and $B$, the set $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ is called the Cartesian product of $A$ and $B$.

Definition 35 For any sets $A$ and $B$, a subset $R$ of $A \times B$ is called a relation from $A$ to $B$. If $(a, b) \in R$, we say $a$ is related to $b$.

Definition 36 For any sets $X$ and $Y$, a relation from $X$ to $Y$ in which every element $x$ of $X$ is related to a unique element $y$ of $Y$ is called a function. Let $f : X \to Y$ denote a function from $X$ to $Y$. If $x$ is related to $y$ we write $f(x) = y$, and refer to $y$ as the image of $x$ under $f$. Also, $X$ is called the domain of $f$ and $Y$ the codomain of $f$.

Note: A function $f : X \to Y$ is also referred to as a mapping, and we say $f$ maps $X$ to $Y$. Also, we say $f$ maps $x$ to $y$ if $f(x) = y$.

Definition 37 Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$. The set $f(A) = \{y \in Y \mid y = f(x), x \in A\}$ is called the image of $A$ under $f$, and the set $f^{-1}(B) = \{x \in X \mid f(x) = y \in B\}$ is called the preimage of $B$. In particular, $f(X)$ is called the range of $f$.

Examples Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then,

\[
\begin{align*}
f(-1) &= 1 \\
f([-2, 1]) &= [0, 2] \\
f(\mathbb{R}) &= [0, \infty) \\
f^{-1}(-1) &= \emptyset \\
f^{-1}(1) &= \{-1, 1\} \\
f^{-1}([-2, 1]) &= [0, 1]
\end{align*}
\]

Definition 38 Let $X$, $Y$ be topological spaces, and $V \subseteq Y$. If $f : X \to Y$, and $V$ open in $Y$ implies $f^{-1}(V)$ is open in $X$, then $f$ is globally continuous.

Note: When $f$ is globally continuous, we often simply say “$f$ is continuous.”

Theorem 39 Let $X$, $Y$ be topological spaces, $V \subseteq Y$ and $f : X \to Y$. Then $f$ is continuous if and only if $V$ closed in $Y$ implies $f^{-1}(V)$ is closed in $X$. 

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Proof. \((\rightarrow)\) Let \(f : X \to Y\) be continuous. Then for an arbitrary open set \(V \subseteq Y\), \(f^{-1}(V)\) is open in \(X\). Now, \(V^C\) is closed in \(Y\), and \(f^{-1}(V^C) = (f^{-1}(V))^C\) is also closed, thus proving our result.

\((\leftarrow)\) Let \(V\) be open in \(Y\). Then \(V^C\) is closed in \(Y\) by definition. Since \(V^C\) closed in \(Y\) implies \(f^{-1}(V^C) = (f^{-1}(V))^C\) is closed in \(X\) by our hypothesis, we know \(f^{-1}(V)\) is open in \(X\). Because this fits the definition of continuity, our result is proven. ■

**Theorem 40** A function \(f : X \to Y\) is continuous if and only if for every \(x \in X\) and every open set \(U\) containing \(f(x)\), there exists a neighborhood \(V\) of \(x\) such that \(f(V) \subseteq U\).

**Proof.** \((\rightarrow)\) Let \(f : X \to Y\) be continuous. Consider a point \(x \in X\) and an open set \(U \subseteq Y\) which contains \(f(x)\). Set \(V = f^{-1}(U)\). Clearly, \(f(x) \in U\) implies \(x \in f^{-1}(U) = V\). Furthermore, \(V\) is open in \(X\) by definition of continuity. Hence, \(V\) is a neighborhood of \(x\). Finally, \(V = f^{-1}(U)\) implies \(f(V) = f(f^{-1}(U)) \subseteq U\), thus proving the desired result.

\((\leftarrow)\) Suppose for every \(x \in X\) and every open set \(U\) containing \(f(x)\), there exists a neighborhood \(V\) of \(x\) such that \(f(V) \subseteq U\). Now, let \(W\) be an arbitrary open set in \(Y\). Choose an arbitrary \(x \in f^{-1}(W) \subseteq X\). Then \(x \in f^{-1}(W)\) implies that \(f(x) \in f(f^{-1}(W)) \subseteq W\). By hypothesis, there exists a neighborhood \(V_x\) of \(x\) in \(X\) such that \(f(V_x) \subseteq W\), which consequently tells us that \(V_x \subseteq f^{-1}(W)\). Thus we have shown that for an arbitrary \(x\) in \(f^{-1}(W)\) there is an open set \(V_x\) such that \(x \in V_x \subseteq f^{-1}(W)\). By **Theorem 6**, \(f^{-1}(W)\) is open in \(X\), thus completing the proof. ■

We now have the necessary tools for the classical ”\(\delta – \epsilon\)” definition of continuity.

**Definition 41** A function \(f : \mathbb{R} \to \mathbb{R}\) is continuous at \(x_0\) if for every \(\epsilon > 0\), there exists a \(\delta > 0\) such that for all \(x \in \mathbb{R}\), \(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon\).

**Theorem 42** A function \(f : \mathbb{R} \to \mathbb{R}\) is continuous if and only if it is continuous at each point \(x_0 \in \mathbb{R}\).

**Proof.** \((\rightarrow)\) Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous. **Theorem 40** tells us that for each \(x \in \mathbb{R}\) and every open set \(U\) containing \(f(x)\), there exists a neighborhood \(V\) of \(x\) such that \(f(V) \subseteq U\). Thus, for each \(x_0 \in \mathbb{R}\) and each \(\epsilon > 0\), consider the open interval \((f(x_0) - \epsilon, f(x_0) + \epsilon)\). Then \(V\) is a union of open intervals, at least one of which contains \(x_0\), and \(f(V) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)\).
Consider an arbitrary open interval of $V$ containing $x_0$ - namely, $(x_0 - \delta_\alpha, x_0 + \delta_\beta); \delta_\alpha, \delta_\beta > 0$. Let $\delta = \min \{\delta_\alpha, \delta_\beta\}$. Then $(x_0 - \delta, x_0 + \delta) \subseteq (x_0 - \delta_\alpha, x_0 + \delta_\beta) \subseteq V$, which implies that $f((x_0 - \delta, x_0 + \delta)) \subseteq f(V) \subseteq U$. It follows that if $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow |x - x_0| < \delta$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \Rightarrow |f(x) - f(x_0)| < \epsilon$, at which point we have arrived at the desired result.

$(\leftarrow)$ Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at every point $x_0 \in \mathbb{R}$. By **Definition 41**, we know that for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $x \in \mathbb{R}$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Let $U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ and $V = (x_0 - \delta, x_0 + \delta)$. Now, $f(x) \in f(V)$ tells us that $x \in V$, which implies $|x - x_0| < \delta$. By hypothesis, this tells us that $|f(x) - f(x_0)| < \epsilon$, which implies $f(x) \in U$. Hence, $f(x) \in f(V)$ implies $f(x) \in U$. It follows that $f(V) \subseteq U$. Thus, we may conclude from **Theorem 40** that $f$ is continuous. ■

**Theorem 43** Let $X, Y$ be topological spaces. Define $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$ and some fixed $y_0 \in Y$. Then $f$ is continuous.

**Proof.** Consider some open set $V \subseteq Y$. Suppose $y_0 \in V$. Then $f^{-1}(V) = X$, which is open. Now, suppose $y_0 \notin V$. Then $f^{-1}(V) = \emptyset$, which is also open. In either case, $f^{-1}(V)$ is open in $X$, so $f$ is continuous. ■

**Theorem 44** The identity function $id : (X, \mathcal{T}) \to (X, \mathcal{T})$, given by $id(x) = x$, is continuous.

**Proof.** Let $A$ be an open set in $X$. Then $id^{-1}(A) = id(A) = A$ is open by assumption. Thus, $id$ is continuous. ■

**Theorem 45** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $c \in \mathbb{R}$. Then $cf : \mathbb{R} \to \mathbb{R}$ defined by $(cf)(x) = c \cdot f(x)$ is continuous.

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $c \in \mathbb{R}$. Suppose $c = 0$. Then $c \cdot f(x) = 0$ for all $x$. Since this defines a constant function, it is continuous by **Theorem 43**.

Suppose $c \neq 0$. Since $f$ is continuous, for each $x_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Then for each $c \in \mathbb{R} - \{0\}$, $|c| \epsilon > 0$, and

$$
|c| \epsilon > |c| |f(x) - f(x_0)| \\
= |c[f(x) - f(x_0)]| \\
= |c \cdot f(x) - c \cdot f(x_0)|,
$$

which is precisely the result we desired. Hence, $cf$ is continuous. ■
Theorem 46 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $(f + g) : \mathbb{R} \to \mathbb{R}$, defined by $(f + g)(x) = f(x) + g(x)$, is continuous.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then for each $x_0 \in \mathbb{R}$ and each $\epsilon_1, \epsilon_2 > 0$, there exists a $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \epsilon_1$ and $|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \epsilon_2$. Now, let $\epsilon = \epsilon_1 + \epsilon_2 > 0$, and set $\delta = \min \{\delta_1, \delta_2\}$. Suppose $\delta > |x - x_0|$. It follows that $\delta_1 \geq \delta > |x - x_0|$ and $\delta_2 \geq \delta > |x - x_0|$. Since $f$ and $g$ are continuous, $\epsilon_1 > |f(x) - f(x_0)|$ and $\epsilon_2 > |g(x) - g(x_0)|$. Adding these inequalities, we have

$$\epsilon_1 + \epsilon_2 = \epsilon > |f(x) - f(x_0)| + |g(x) - g(x_0)| \\
\geq |f(x) - f(x_0) + g(x) - g(x_0)| \text{, by Triangle Inequality} \\
= ||f(x) + g(x)| - |f(x_0) + g(x_0)|| \\
= |(f + g)(x) - (f + g)(x_0)|.$$

Since $x_0$ was arbitrary, $(f + g)(x)$ is continuous. ■

Theorem 47 Let $X, Y$, and $Z$ be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composition $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$, is continuous.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous. Consider an open set $A \subseteq Z$. Since $g$ is continuous, $g^{-1}(A)$ is open in $Y$, and since $f$ is continuous, $f^{-1}(g^{-1}(A))$ is open in $X$. Thus, for an arbitrary set $A \subseteq Z$, $A$ open in $Z$ implies $(g \circ f)^{-1}(A)$ is open in $X$. Thus, $g \circ f$ is continuous. ■

2.6 Homeomorphism

Definition 48 A function $f : X \to Y$ is injective if, for all $x_1, x_2 \in X$ such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Definition 49 A function $f : X \to Y$ is surjective if for all $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. 

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Definition 50  A function \( f : X \to Y \) is **bijective**, or a **bijection**, if it is both injective and surjective.

**Note:** We may also call a bijective function \( f \) a one-to-one correspondence from \( X \) to \( Y \), since \( f \) matches every point \( x \in X \) to a unique point \( y \in Y \).

**Example 51** The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is not injective, since \( f(-1) = f(1) \), but \(-1 \neq 1 \), nor is \( f \) surjective, since no \( x \in \mathbb{R} \) exists such that \( f(x) = -1 \).

**Example 52** The function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = e^x \) is injective but not surjective, since no \( x \in \mathbb{R} \) exists such that \( f(x) = 0 \).

**Example 53** The function \( h : \mathbb{R} \to \mathbb{R} \) defined by \( h(x) = x^3 - x \) is surjective but not injective, since \( f(0) = f(1) \) but \( 0 \neq 1 \).

**Example 54** The function \( j : \mathbb{R} \to \mathbb{R} \) defined by \( j(x) = 3x - 7 \) is both injective and surjective. Thus, \( j \) is bijective by definition.

**Definition 55** Let \( f : X \to Y \) be bijective. Given \( x \in X \), let \( y \in Y \) be the unique point such that \( f(x) = y \). Then the **inverse** of \( f \), denoted \( f^{-1} \), is the function \( f^{-1} : Y \to X \) defined by \( f^{-1}(y) = x \).

**Note:** Do not confuse the preimage \( f^{-1} \) and the inverse function \( f^{-1} : Y \to X \). When either of these ideas appear in the paper, it will be made clear from the context which concept we are discussing.

**Theorem 56** Let \( f : X \to Y \) be bijective, and let \( g : Y \to X \). Then \( f^{-1} = g \) if and only if \( f \circ g = g \circ f = \text{id} \).

**Proof.** (\( \to \)) Suppose \( g = f^{-1} \). Then let \( x \in X \) and let \( y \in Y \) be the unique point such that \( f(x) = y \). By definition, \( f^{-1}(y) = x \). Now,

\[
(g \circ f)(x) = g(f(x)) \\
= f^{-1}(f(x)) \\
= f^{-1}(y) \\
= x,
\]
and
\[(f \circ g)(y) = f(g(y)) = f(f^{-1}(y)) = f(x) = y.\]

Hence, \(f \circ g = g \circ f = \text{id}.\)

\((\leftarrow)\) Suppose \(f \circ g = g \circ f = \text{id}.\) If \(x \in X\) and \(y \in Y\) is the unique point such that \(f(x) = y\), then
\[g(y) = g(f(x)) = (g \circ f)(x) = x.\]

Hence, by definition, \(g = f^{-1}.\) ■

**Theorem 57** A function \(f : X \to Y\) is bijective if and only if an inverse function \(f^{-1} : Y \to X\) exists.

**Proof.** \((\rightarrow)\) Let \(f : X \to Y\) be bijective. Then \(f\) is surjective, so for all \(y \in Y\) there exists an \(x \in X\) such that \(f(x) = y\). Also, \(f\) is injective, so \(x\) is unique. Thus, let \(g : Y \to X\) be defined by \(g(y) = x\). then \(g(f(x)) = g(y) = x\), and \(f(g(y)) = f(x) = y\). By **Theorem 56**, \(g = f^{-1},\) so \(f\) has an inverse.

\((\leftarrow)\) Let \(f : X \to Y\) have an inverse function \(f^{-1} : Y \to X.\) Now, suppose \(f(a) = f(b)\). Then \(f^{-1}(f(a)) = f^{-1}(f(b))\) which implies \(a = b\). Thus, \(f\) is injective. Let \(y \in Y\) and let \(f^{-1}(y) = x\) for some \(x \in X.\) Then \(y = f(f^{-1}(y)) = f(x),\) which tells us that for every \(y \in Y\) there exists an \(x \in X\) such that \(y = f(x)\). Hence, \(f\) is surjective. Combined with our earlier result, \(f\) is bijective. ■

**Definition 58** Let \(X, Y\) be topological spaces. Consider a bijective function \(f : X \to Y\) and its inverse, \(f^{-1} : Y \to X.\) Then \(f\) is a **homeomorphism** if \(f\) and \(f^{-1}\) are both continuous. If such a function \(f\) exists, then \(X\) and \(Y\) are said to be **homeomorphic** or **topologically equivalent**, in which case we write \(X \cong Y.\)
Consider a homeomorphism $f : X \to Y$ and its inverse $f^{-1} : Y \to X$. Since $f^{-1}$ is continuous, $(f^{-1})^{-1}(U)$ is open in $Y$ for every open set $U$ in $X$. But, because homeomorphisms are bijective, $(f^{-1})^{-1}(U) = f(U)$ for all $U \subseteq X$. Similarly, since $f$ is continuous, $f^{-1}(V)$ is open in $X$ for every open set $V \subset Y$. Thus, a homeomorphism $f$ not only matches every point $x$ to a unique point $y$, but every open set $U \subseteq X$ to a unique open set $V \subseteq Y$. As an example of this, define $f : \{a, b, c\} \to \{1, 2, 3\}$ by $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$, and give the sets $\{a, b, c\}$ and $\{1, 2, 3\}$ the topologies as seen in Figure 6. Notice that not only is there a one-to-one correspondence between points, but the fact that $f(\{a, b\}) = \{2, 3\}$ shows that a one-to-one correspondence exists between the open sets as well. Thus $f$ is a homeomorphism.

Figure 6: The function $f : \{A, B, C\} \to \{1, 2, 3\}$ is a homeomorphism.
2.7 Continuous Functions and Connectedness; The Intermediate Value Theorem

To prove that continuous functions preserve connectedness, we will first need a brief lemma which allows us to separate unions and intersections in the preimage of a function.

**Lemma 59** Let \( f : X \to Y \), and let \( A \) and \( B \) be subsets of \( Y \). Then,

1. \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \)
2. \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \).

**Proof.** We shall proceed with the proof of Part 1. Let \( f : X \to Y \) and \( A \) and \( B \) be subsets of \( Y \). Now, consider \( y \in A \cup B \) and \( x \in f^{-1}(A \cup B) \) such that \( f^{-1}(y) = x \). Then,

\[
x \in f^{-1}(A \cup B) \iff y \in A \cup B \iff y \in A \lor y \in B \iff (f^{-1}(y) \in f^{-1}(A)) \lor (f^{-1}(y) \in f^{-1}(B)) \iff f^{-1}(y) \in f^{-1}(A) \cup f^{-1}(B) \iff x \in f^{-1}(A) \cup f^{-1}(B).
\]

The proof of Part 2 is similar. \( \blacksquare \)

**Theorem 60** Let \( X, Y \) be topological spaces, and let \( f : X \to Y \) be continuous. If \( A \) is a connected subset of \( X \), then \( f(A) \) is a connected subset of \( Y \).

**Proof.** We will prove the contrapositive. Let \( X, Y \) be topological spaces, \( f : X \to Y \) be continuous, and \( A \) be a subset of \( X \). If \( f(A) \) is a disconnected subset of \( Y \), then, by **Lemma 31**, there exist disjoint, open subsets \( U' \) and \( V' \) of \( Y \) such that \( f(A) \subseteq U' \cup V' \), \( U' \cap V' \subseteq f(A)^C \), \( U' \cap f(A) \neq \emptyset \), and \( V' \cap f(A) \neq \emptyset \). Since \( f \) is continuous, \( f^{-1}(U') = U \) and \( f^{-1}(V') = V \) are open subsets of \( X \). Now,

\[
A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(U' \cup V') = f^{-1}(U') \cup f^{-1}(V'), \quad \text{by Lemma 59: Part 1}
\]
\[
= U \cup V,
\]

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and,

\[ U \cap V = f^{-1}(U') \cap f^{-1}(V') \]
\[ = f^{-1}(U' \cap V'), \quad \text{by Lemma 59: Part 2} \]
\[ \subseteq f^{-1}(f(A^c)) \]
\[ = f^{-1}(f(A))^c \]
\[ \subseteq A^c. \]

Also, since \( U' \cap f(A) \neq \emptyset \),

\[ f^{-1}(U' \cap f(A)) \neq \emptyset \implies f^{-1}(U') \cap f^{-1}(f(A)) \neq \emptyset \]
\[ \implies U \cap A \neq \emptyset. \]

Similarly, \( V \cap A \neq \emptyset \). It follows from Lemma 31 that \( A \) is a disconnected subset of \( X \), thus completing our proof. □

**Definition 61** A set \( A \subseteq \mathbb{R} \) is an interval if \( A \) contains at least two distinct points, and if \( a, b \in \mathbb{R} \) such that \( a < b \), then every point \( x \) such that \( a < x < b \) lies in \( A \).

**Examples** \( (-\infty, a), (a, b], (a, b), [a, b], \) and \( (-\infty, \infty) \) are a few examples of intervals.

**Theorem 62** Let \( A \subseteq \mathbb{R} \) with at least two distinct points. Then \( A \) is connected if and only if \( A \) is an interval.

**Proof.** (\( \rightarrow \)) We shall prove the contrapositive. Suppose \( A \subseteq \mathbb{R} \) is not an interval. Then, by **Definition 61**, there must exist points \( a, b, c \in \mathbb{R} \) such that \( a < c < b \) with \( a, b \in A \) but \( c \notin A \). Now, consider the open intervals \( B = (-\infty, c) \) and \( C = (c, \infty) \). Clearly, \( A \subseteq B \cup C \), \( B \cap C = \emptyset \subseteq A^c \), \( a \in A \cap B \neq \emptyset \) and \( b \in A \cap C \neq \emptyset \). Consequently, \( A \) is disconnected by Lemma 31.

(\( \leftarrow \)) Again, we will prove the contrapositive. Suppose \( A \subseteq \mathbb{R} \) is disconnected. Then by **Lemma 33**, there exist closed sets \( U \) and \( V \) in \( \mathbb{R} \) such that \( A \subseteq U \cup V \), \( U \cap V \subseteq A^c \), \( U \cap A \neq \emptyset \), and \( V \cap A \neq \emptyset \). Note that the last two conditions say that there exists at least one point \( u \in U \cap A \) and at least one point \( v \in V \cap A \). Suppose \( u = v \). Then \( u \in A \) and
\( u \in U \cap V \subseteq A^C \), which is a contradiction. Thus, \( u < v \) or \( u > v \). Without loss of generality, we may assume \( u < v \).

Consider \( V' = V \cap [u, v] \neq \emptyset \), which is closed by Theorem 13:C3. Note that \( V' \) is bounded below by \( u \), and has a greatest lower bound \( c \) by Axiom 10. By Theorem 22, \( V' \) must contain this boundary point \( c \). Notice that \( c \neq u \), since \( c \in V \) and \( u \in U \), which means \( u \in U \cap V \subseteq A^C \) and \( u \in A \), which is impossible. Thus, \( c > u \).

Now, consider \( U' = U \cap [u, c] \neq \emptyset \), which is also closed. \( U' \) is bounded above by \( c \), so it must have a least upper bound \( d \) and must contain it. Suppose \( c = d \). Then \( c \in U \cap V \), which implies that \( c \in A^C \). Hence, \( c \) is a point between two points in \( A \) - namely, \( u \) and \( v \)- but which itself is not in \( A \), so \( A \) is not an interval. But, if \( d < c \), then the open interval \((d, c)\) is neither in \( U \), nor is it in \( V \), so \((d, c) \cap (U \cup V) = \emptyset \), which tells us \((d, c) \cap A = \emptyset \), meaning the points \((d, c)\) are all between \( u \) and \( v \) but are not in \( A \), so \( A \) is yet again not an interval. It follows by contrapositive that intervals in \( \mathbb{R} \) are connected. ■

**Theorem 63 (Intermediate Value Theorem)** Suppose \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be continuous and \( f(a) \neq f(b) \). Then for each point \( y \in [f(a), f(b)] \) there exists a point \( c \in [a, b] \) such that \( y = f(c) \) (see Figure 7).

![Figure 7: There is a point \( c \in [a, b] \) such that \( y = f(c) \in [f(a), f(b)] \).](image)
**Proof.** Let $a, b \in \mathbb{R}$, $a < b$, and suppose $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) \neq f(b)$. The interval $[a, b]$ is connected, so $f([a, b])$ is also connected and, therefore, an interval. Now, $f(a)$ and $f(b)$ are clearly elements of $f([a, b])$. Because $f([a, b])$ is an interval, there exists some point $y$ between $f(a)$ and $f(b)$ such that $y \in f([a, b])$. Consequently, there exists a point $c \in [a, b]$ such that $y = f(c)$. ■

**Corollary 64** Let $f : [a, b] \to \mathbb{R}$, $a < b$. If $f(a)$ and $f(b)$ have opposite signs, there exists a point $c \in [a, b]$ such that $f(c) = 0$.

**Proof.** Suppose $f(a)$ and $f(b)$ have opposite signs. Without loss of generality, assume $f(a) < 0 < f(b)$. By Theorem 63, there is a point $c \in [a, b]$ such that $f(c) = 0$. ■

### 2.8 The Fixed-Point Theorem

**Definition 65** Let $X$ be a topological space and $f : X \to X$. A **fixed point** of $f$ is a point $x \in X$ such that $f(x) = x$.

**Theorem 66 (One-Dimensional Brouwer Fixed Point Theorem)** Every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

**Proof.** Let $f : [0, 1] \to [0, 1]$ be continuous. The trivial cases where $f(0) = 0$ or $f(1) = 1$ obviously satisfy the conclusion. Therefore, we will consider only functions $f$ such that $f(0) > 0$ and $f(1) < 1$. Define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = x - f(x).$$

Since $f(x)$ is continuous on $[0, 1]$ by hypothesis, and the identity function $id(x) = x$ is continuous on $[0, 1]$ by Theorem 44, $g(x)$ is also continuous on $[0, 1]$ by Theorems 45 and 46. Now, if we evaluate $g$ at 0 and 1, we have

$$g(0) = 0 - f(0) = -f(0) < 0$$

and,

$$g(1) = 1 - f(1) > 1 - 1 = 0.$$ 

Thus, $g(0)$ is negative, and $g(1)$ is positive. It follows from Corollary 64 that there exists a point $z \in [0, 1]$ such that $g(z) = f(z) - z = 0$. Consequently, $f(z) = z$, which proves that $z$ is precisely the fixed point we required. ■
3 Two-Dimensional Case of the Brouwer Fixed-Point Theorem

Although the two-dimensional case is not quite as apparent as the one-dimensional case, anyone who has ever been lost at an amusement park should be familiar with it. Consider a function \( f \) which assigns every point in the amusement park to a point on a map of the park. We can think of \( f \) as a continuous function from the park to itself. Assuming the park is topologically equivalent to a two-dimensional disk, the two-dimensional case of the Brouwer Fixed-Point Theorem states that \( f \) has a fixed point. On a map posted on the ground at a stationary location in the park, this fixed point is usually marked with a yellow star, as well as a caption which says “You are here!”

3.1 Product Spaces; The Standard Topology on \( \mathbb{R}^n \)

Theorem 67 Let \( A, B, C, \) and \( D \) be sets. Then \((A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)\).

Proof. Recalling the definition of a Cartesian product of sets (Definition 34), an ordered pair \((i, j)\) is an element of \( I \times J \) if and only if \( i \in I \) and \( j \in J \). Thus,

\[
(x, y) \in (A \cap B) \times (C \cap D) \iff (x \in A \cap B) \land (y \in C \cap D) \\
\iff (x \in A \land x \in B) \land (y \in C \land y \in D) \\
\iff (x \in A \land y \in C) \land (x \in B \land y \in D) \\
\iff ((x, y) \in A \times C) \land ((x, y) \in B \times D) \\
\iff (x, y) \in (A \times C) \cap (B \times D).
\]

Thus, \((A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)\). \(\blacksquare\)

Theorem 68 Let \( X \) and \( Y \) be topological spaces. Then the collection of sets

\[ B = \{ U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y \} \]

is a basis for a topology on \( X \times Y \).
Proof. We show that $\mathcal{B}$ satisfies the axioms for a basis.

B1. Let $x \in X$ and $y \in Y$. Since $X$ and $Y$ are open, it follows that for every $(x, y) \in X \times Y$ there is a set $B_X \times B_Y \in \mathcal{B}$ such that $(x, y) \in \mathcal{B}$, i.e., $B_X = X$ and $B_Y = Y$.

B2. Suppose for open sets $U_1$, $U_2$ in $X$ and $V_1$, $V_2$ in $Y$, $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$ and $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. We must show that there exists a set $U_3 \times V_3 \in \mathcal{B}$ such that $(x, y) \in U_3 \times V_3 \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$. Let $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Certainly, $U_3$ is open in $X$ and $V_3$ is open in $Y$. Thus, $U_3 \times V_3 \in \mathcal{B}$. Now, since $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$, it follows that $x \in U_1$ and $x \in U_2$, which implies that $x \in U_1 \cap U_2 = U_3$. Similarly, $y \in V_1 \cap V_2 = V_3$. Consequently, $(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2) = U_3 \times V_3$. Finally, by Theorem 67, $U_3 \times V_3 = (U_1 \times V_1) \cap (U_2 \times V_2)$, which is a subset of itself. Thus, $U_3 \times V_3$ satisfies the required axioms, and our proof is complete. ■

Definition 69 For topological spaces $X$ and $Y$, the \textit{product topology over $X \times Y$} is the topology generated by the collection of sets $\mathcal{B}$ proven to be a basis in Theorem 68.

Definition 70 The set $\mathbb{R}^2$, which denotes \textit{Euclidean 2-space}, also called \textit{the plane}, is the Cartesian product $\mathbb{R} \times \mathbb{R}$. Further, $\mathbb{R}^n$, which denotes \textit{Euclidean $n$-space}, is the Cartesian product of $n$ copies of $\mathbb{R}$. The point $(0, 0, ..., 0) \in \mathbb{R}^n$ is called the \textit{origin}.

Definition 71 Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be points in $\mathbb{R}^n$. Define the \textit{Euclidean distance from $x$ to $y$}, also called the \textit{Euclidean metric}, or simply distance, to be

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}.$$ 

Note that, by definition of the square root function, $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$.

Definition 72 The \textit{$n$-ball}, denoted $B^n$, is the set of all points which lie a distance of 1 or less from the origin in $\mathbb{R}^n$. Stated another way,

$$B^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\}.$$ 

Further, the \textit{open $n$-ball}, denoted $B^n$ is the set of all points in $\mathbb{R}^n$ which lie a distance less than 1 from the origin,

$$B^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 < 1\}.$$ 

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Note: $B^2$ is often called the disk, and $\mathring{B}^2$ is often called the open disk.

**Definition 73** Let $x$ be a point in $\mathbb{R}^n$, and let $\epsilon > 0$. Define the set $\mathring{B} (x, \epsilon)$, called the open ball of radius $\epsilon$ centered at $x$, by

$$\mathring{B} (x, \epsilon) = \{ y \in \mathbb{R}^n \mid d(x, y) < \epsilon \}.$$ 

We will now introduce a brief lemma which will allow us to define the standard topology on $\mathbb{R}^n$.

**Lemma 74** Let $x \in \mathbb{R}^n$ and let $r > 0$. Then, for every $p \in \mathring{B} (x, r)$, there exists an $\epsilon > 0$ such that $\mathring{B} (p, \epsilon) \subseteq \mathring{B} (x, r)$ (see Figure 8).

**Proof.** Suppose $p \in \mathring{B} (x, r)$. By definition, $d(p, x) < r$, which implies $0 < r - d(p, x)$. Let $0 < \epsilon < r - d(p, x)$. Suppose $q \in \mathring{B} (p, \epsilon)$. It follows that

$$d(x, q) \leq d(x, p) + d(p, q)$$

$$< d(x, p) + \epsilon$$

$$< d(x, p) + (r - d(p, x)) = r.$$ 

Hence, by definition, $q \in \mathring{B} (x, r)$. Thus, $\mathring{B} (p, \epsilon) \subseteq \mathring{B} (x, r)$, and our proof is complete. ■

![Figure 8: There exists $\epsilon$ such that $\mathring{B} (p, \epsilon) \subseteq \mathring{B} (x, r)$.](image)
Theorem 75  The collection

\[ \mathcal{B} = \left\{ \overset{\circ}{B}(x, \epsilon) \mid x \in \mathbb{R}^n, \epsilon > 0 \right\} \]

is a basis for a topology on \( \mathbb{R}^n \).

Proof. B1. Since \( x \in \overset{\circ}{B}(x, 1) \) for all \( x \in \mathbb{R}^n \), the first condition for a basis is satisfied.

B2. Suppose \( x \in \overset{\circ}{B}(p, r_1) \cap \overset{\circ}{B}(q, r_2) \). Lemma 74 tells us that there exist \( \epsilon_1, \epsilon_2 > 0 \) such that \( \overset{\circ}{B}(x, \epsilon_1) \subseteq \overset{\circ}{B}(p, r_1) \) and \( \overset{\circ}{B}(x, \epsilon_2) \subseteq \overset{\circ}{B}(q, r_2) \). Let \( \epsilon = \min\{\epsilon_1, \epsilon_2\} \). Then,

\[ \overset{\circ}{B}(x, \epsilon) \subseteq \overset{\circ}{B}(x, \epsilon_1) \cap \overset{\circ}{B}(x, \epsilon_2) \subseteq \overset{\circ}{B}(p, r_1) \cap \overset{\circ}{B}(q, r_2). \]

Hence, the second condition for a basis holds.

Since both B1 and B2 were satisfied, \( \mathcal{B} \) is a basis for a topology on \( \mathbb{R}^n \).

Definition 76  The topology generated by the basis \( \mathcal{B} \) from Theorem 75 is called the standard topology on \( \mathbb{R}^n \).

Definition 77  Let \( Y \subseteq \mathbb{R}^n \). The standard topology on \( Y \) is the topology that \( Y \) inherits from the standard topology on \( \mathbb{R}^n \).

3.2 Homotopy

Definition 78  Let \( X \) and \( Y \) be topological spaces, and let \( f, g : X \to Y \) be continuous functions. Assume \( I = [0, 1] \) has the standard topology, and \( X \times I \) has the product topology. Then \( f \) and \( g \) are homotopic if there exists a continuous function \( F : X \times I \to Y \) such that \( F(x, 0) = f(x) \) and \( F(x, 1) = g(x) \). \( F \) is called a homotopy from \( f \) to \( g \), and \( f \simeq g \) denotes that \( f \) and \( g \) are homotopic.

Homotopy is a powerful tool for studying functions. We can use it to continuously “melt” one function into another (see Figure 9), but more importantly, we can use homotopy to sort continuous functions into equivalence classes, since \( \simeq \) happens to be an equivalence relation. We will prove this fact in a moment, but first we need a brief lemma on the continuity of piecewise functions.
Figure 9: The homotopy $F$ continuously “melts” $f$ into $g$.

**Lemma 79 (The Pasting Lemma)** Let $X, Y$ be topological spaces and let $A, B \subseteq X$ be closed sets such that $A \cup B = X$. If $f : A \rightarrow Y$ and $g : B \rightarrow Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \rightarrow Y$ defined by

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continuous.

**Proof.** To prove this result, we will use **Theorem 39** and show that $V$ closed in $Y$ implies $h^{-1}(V)$ is closed in $X$. Assume $V$ is closed in $Y$. Now, $h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$. Since $f$ is continuous, $f^{-1}(V)$ is closed in $A$. By **Theorem 26**, $f^{-1}(V) = D \cap A$ for some set $D$ closed in $X$. Since $D$ and $A$ are both closed in $X$, it follows from **Theorem 13**:C3 that $D \cap A = f^{-1}(V)$ is closed in $X$. Similarly, $g^{-1}(V)$ is closed in $X$. Hence, $h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$ is closed in $X$ by **Theorem 13**:C4.

**Theorem 80** Homotopy is an equivalence relation on the set of all continuous functions $f : X \rightarrow Y$.

**Proof.** To prove that $\simeq$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

**Reflexive:** Let $f : X \rightarrow Y$ be continuous. Define $F : X \times I \rightarrow Y$ by $F(x,t) = f(x)$. Clearly, $F(x,0) = F(x,1) = f(x)$, indicating that $f \simeq f$.

**Symmetric:** Let $f \simeq g$ for continuous functions $f, g : X \rightarrow Y$. Then there exists a homotopy $F : X \times I \rightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Define a function
$G : X \times I \to Y$ by $G(x, t) = F(x, 1 - t)$. Since $1 - t$ is continuous, $G$ is continuous as well. Also, $G(x, 0) = F(x, 1) = g(x)$ and $G(x, 1) = F(x, 0) = f(x)$. Thus, $G$ is a homotopy from $g$ to $f$, meaning $g \simeq f$.

**Transitive:** Let $f, g, h : X \to Y$ be continuous, $f \simeq g$, and $g \simeq h$. Then there exist homotopies $F$ and $G$. Now, define $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $2t$ and $2t - 1$ are continuous, it follows that $F(x, 2t)$ and $G(x, 2t - 1)$ are also continuous. At $t = \frac{1}{2}$, $F(x, 2t) = G(x, 2t - 1) = g(x)$, so the Pasting Lemma (Lemma 79) tells us that $H$ is continuous. Now, $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$, meaning $H$ is a homotopy from $f$ to $h$. Hence, $f \simeq h$. ■

**Definition 81** Let $C(X, Y)$ denote the set of all continuous functions $f : X \to Y$. The equivalence classes under $\simeq$ are called homotopy classes in $C(X, Y)$. The homotopy class containing a function $f$ is denoted $[f]$.

To state Definition 81 a different way, $[f]$ denotes the set of all functions in $C(X, Y)$ which are homotopic to $f$. Upon first glance, the ability to create homotopy classes may seem pointless at best. For example, all continuous functions $f : \mathbb{R} \to \mathbb{R}$ are homotopic to each other, since they are all homotopic to $f_0(x) = 0$ by the homotopy $F(x, t) = (1 - t)f(x)$. However, the ability to distinguish homotopy classes becomes invaluable when we move on to circle functions, which we will now define, along with some useful terminology.

### 3.3 Circle Functions and Degree

**Definition 82** The $n$-sphere, denoted $S^n$, is the set of all points which lie a distance of 1 from the origin in $\mathbb{R}^{n+1}$. Put another way, $S^n$ is the boundary of $B^{n-1}$.

**Remark 83** $S^1$ is often called the circle. Based on Definitions 76 and 77, as well as Theorem 25, the standard topology on the circle is generated by open intervals in $S^1$. These open intervals consist of all points on the circle between two radii (see Figure 10).

**Definition 84** A continuous function $f : S^1 \to S^1$ is called a circle function.
Figure 10: The standard basis on $S^1$ is the intersection of $S^1$ with open balls in $\mathbb{R}^2$.

To represent points on $B^2$, it is natural to use polar coordinates $[r, \theta]$, where $r$ is the distance from the origin, and $\theta$ is the usual angle measured from the positive $x$-axis in the plane. When considering circle functions, however, it is convenient to represent points on $S^1$ simply by using $\theta$, since $r = 1$ for all points on $S^1$. Further, the point $\theta = 0$ is a distinguished point in $S^1$, denoted by $\ast$, and is called the base point.

An important aspect of circle functions is the number of times the function “wraps the circle around itself.” For example, $f(\theta) = \theta$ will hit every point on the circle exactly one time, and so wraps itself around the circle once. Similarly, $f(\theta) = 2\theta$ wraps itself around $S^1$ twice, and we say that $f(\theta) = -\theta$ wraps itself around the circle “$-1$ times,” or one time in the negative direction. Generally, $c_n(\theta) = n\theta$, $n \in \mathbb{Z}$ wraps itself around $S^1$ exactly $n$ times.

Unfortunately, not all circle functions are as nice as $c_n(\theta)$. Circle functions can oscillate, backtrack, and pause as they wrap around $S^1$. The question now becomes: is there a way to “deform away all of the complicated behavior via homotopy?” [1, p.278] As a matter of fact, there is such a way, by using our next theorem. The proof of Theorem 85 is quite complicated, though, and uses a number of results which we shall not use at any point afterward. Thus, to maintain organization and forward motion, the proof has been relegated to the appendices.

**Theorem 85** Let $f : S^1 \to S^1$ be a circle function. Then there exists a unique $n \in \mathbb{Z}$ such that $f \simeq c_n$.

**Proof.** See Appendix A.
Definition 86 Let \( f : S^1 \to S^1 \) be a circle function. Then the **degree** of \( f \), denoted by \( \deg(f) \), is defined to be the unique \( n \) such that \( f \simeq c_n \). We refer to the function \( c_n : S^1 \to S^1 \) as the **nth degree function**.

Combining Theorems 85 and 86, it becomes very plain that the homotopy classes of \( C(S^1, S^1) \) are represented by the \( n \)th degree functions \( c_n(\theta) \).

**Theorem 87** Two circle functions \( f \) and \( g \) are homotopic if and only if \( \deg(f) = \deg(g) \).

**Proof.** Let \( f, g \) be homotopic circle functions. Since \( \simeq \) is an equivalence relation, the distinct homotopy classes of \( C(S^1, S^1) \) are disjoint sets. By Theorem 85, we know that the homotopy classes are given by \( \{[c_n] \mid n \in \mathbb{Z}\} \). Hence, \( f \simeq g \) if and only if \( [f] = [g] = [c_n] \), which occurs exactly when \( \deg(f) = \deg(g) = n \) by Definition 86.

### 3.4 Two-Dimensional No-Retraction Theorem

**Definition 88** Let \( X, Y \) be topological spaces, \( A \subseteq X \), and \( f : A \to Y \) be continuous. A **continuous extension of \( f \) onto \( X \)** is a continuous function \( F : X \to Y \) such that \( F(x) = f(x) \) for all \( x \in A \). We say that \( f \) **extends to** \( F \).

**Theorem 89** Consider a circle function \( f : S^1 \to S^1 \). Then \( \deg(f) = 0 \) if and only if there is a continuous extension \( F : B^2 \to S^1 \) of \( f \) onto the disk.

**Proof.** As was stated earlier, we will represent points in \( B^2 \) with polar coordinates \([r, \theta]\).

(\(\to\)) Suppose \( f : S^1 \to S^1 \) has degree 0. **Definition 86** tells us that there exists a homotopy \( G : S^1 \times [0, 1] \to S^1 \) such that \( G(\theta, 0) = c_0(\theta) = * \in S^1 \) and \( G(\theta, 1) = f(\theta) \). Now, let \( F : B^2 \to S^1 \) be defined by \( F[r, \theta] = G(\theta, r) \). Notice that, since \( G(\theta, r) = * \) for all \( \theta \in S^1 \) when \( r = 0 \), \( F \) is well-defined at \( r = 0 \), so \( F \) is defined as a function on \( B^2 \). Also, \( G \) being continuous implies \( F \) is continuous as well. Finally, \( F[1, \theta] = G(\theta, 1) = f(\theta) \), so that \( F \) is a continuous extension of \( f \).

(\(\leftarrow\)) Suppose \( f : S^1 \to S^1 \) extends to a continuous function \( F : B^2 \to S^1 \). Define a function \( G : S^1 \times [0, 1] \to S^1 \) by \( G(\theta, t) = F[t, \theta] \). \( F \) is continuous, and \( G \) is also continuous. By definition, \( G \) is a homotopy between the circle functions \( G(\theta, 0) \) and \( G(\theta, 1) \). **Theorem 87** then tells us that \( G(\theta, 0) \) and \( G(\theta, 1) \) must have the same degree. But, \( G(\theta, 0) = F[0, \theta] \),
which is a constant function and so has degree 0. Hence,
\( G(\theta, 1) = F[1, \theta] = f(\theta) \) also has degree 0. ■

We will need one more very important definition before we tackle the next theorem, from which the Two-Dimensional Brouwer Fixed-Point Theorem follows as a direct consequence.

**Definition 90** Let \( X \) be a topological space and \( A \subseteq X \). A **retraction** from \( X \) onto \( A \) is a continuous function \( r : X \rightarrow A \) such that \( r(a) = a \) for each \( a \in A \). The subset \( A \) is said to be a **retract** of \( X \) if a retraction \( r \) exists.

Put another way, \( r \) is a retraction from \( X \) to \( A \) if \( r \) continuously maps every point in \( X \) to a point in \( A \) while fixing the points already in \( A \).

**Theorem 91 (Two-Dimensional No-Retraction Theorem)** There exists no retraction from the disk \( B^2 \) to its boundary \( S^1 \).

**Proof.** Let us suppose that a retraction \( r : B^2 \rightarrow S^1 \) exists. Then \( r \) is continuous, and \( r[1, \theta] = [1, \theta] \) for all \( \theta \in S^1 \). Now, \( r[1, \theta] = \theta = \text{id}(\theta) \) has degree 1. But, by **Definition 88**, \( r \) is also a continuous extension of \( \text{id} \) onto \( B^2 \) and has degree 0 by **Theorem 89**. Since \( r \) can not have both degree 0 and degree 1, this is a contradiction. Hence, no such continuous function \( r \) can exist. ■

The No-Retraction Theorem is an intuitively clear result. Imagine trying to deform the skin of a drum to the rim without tearing the skin (see Figure 11). The reader will consider for a moment and then declare the act impossible. So too is the act of retracting \( B^2 \) to \( S^1 \).

![Figure 11](image.png)

"Deforming the skin of a drum to the rim will tear it, demonstrating that there is no retraction from \( B^2 \) onto \( S^1 \)." [1, p. 282]
3.5 Fixed-Point Theorem; Fixed-Point Property as a Topological Property

Theorem 92 (Two-Dimensional Brouwer Fixed-Point Theorem) Every continuous function $f : B^2 \to B^2$ has a fixed point.

Proof. Suppose there exists a continuous function $f : B^2 \to B^2$ with no fixed points. Now, consider the ray in $\mathbb{R}^2$ which runs from $f(x)$ through $x$. Because $f$ has no fixed points, $f(x) \neq x$ for every $x \in B^2$, so the ray is well-defined. Now, define $r : B^2 \to S^1$ such that $r(x)$ is the point in $S^1$ which the ray intersects. Thus $r$ maps every point $x \in B^2$ to a point in $S^1$ and $r(x) = x$ when $x \in S^1$ (see Figure 12).

![Figure 12: When $x \neq f(x)$ for all $x \in B^2$, the function $r : B^2 \to S^1$ is well-defined.](image)

Now, let $U$ be open in $S^1$ and let $x$ be a point in $B^2$ such that $r(x) \in U$. We may choose small open balls $O_1$ and $O_2$ centered at $f(x)$ and $x$, respectively, such that every ray which begins in $O_1$ and goes through $O_2$ intersects $S^1$ in $U$. Since $f$ is continuous, by Theorem 40 there is an open set $V$ with $x \in V \subseteq O_2$ such that $f(V) \subseteq O_1$ (see Figure 13). It is clear that for all $v \in V$, the ray starting at $f(v)$ and passing through $v$ intersects $S^1$ in $U$, which tells us $r(V) \subseteq U$. Using Theorem 40 again, $r$ is continuous and, therefore, a retraction. But, this contradicts Theorem 91, so $f$ has a fixed point. ■
Figure 13: There is a neighborhood $V$ of $x$ such that $r(V) \subseteq U$.

Because every continuous function mapping $B^2$ to itself has a fixed point, we say that $B^2$ has the \textit{fixed point property}. We conclude this section by showing that the fixed point property is a \textit{topological property}; i.e., two topological spaces which are topologically equivalent to each other either both have or both lack the fixed point property. Because one of the main goals of topology is to discover what properties are maintained when moving between topologically equivalent spaces, this is an incredibly important proposition to make.

\textbf{Theorem 93} Let $X$ and $Y$ be topological spaces such that $X \cong Y$. If $X$ has the fixed point property, then $Y$ has the fixed point property as well.

\textbf{Proof.} Since $X \cong Y$, there exists a bijective function $f : X \to Y$ such that $f$ and its inverse $f^{-1}$ are continuous. Also, $X$ has the fixed point property, so every continuous function $g : X \to X$ has a fixed point. Now, let $h : Y \to Y$ be continuous. Then $h \circ f : X \to Y$ and, consequently, $f^{-1} \circ h \circ f : X \to X$ are certainly continuous. But, since $X$ has the fixed point property, $f^{-1}(h(f(x))) = x$ for some $x \in X$. It follows that $f(f^{-1}(h(f(x)))) = f(x)$, which implies that $h(f(x)) = f(x)$, proving that $h$ has a fixed point - namely, $f(x)$.

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4 The $n$-Dimensional Case of the Brouwer Fixed-Point Theorem

As we move from two dimensions into $n$ dimensions, we lose the ability to describe the Brouwer Fixed-Point Theorem by way of intuitive examples. In three dimensions, we can think of stirring coffee in a mug. If we consider points to be atoms and assume that stirring is a “continuous function,” the Fixed-Point Theorem states that at every moment the coffee is being stirred, there will be at least one atom located at the same point at which it originated. In four or more dimensions, intuitive examples no longer exist to our three-dimensional minds.

While reading the proof of Theorem 92, the reader may have noticed that the only step firmly anchored in two dimensions is the Two-Dimensional No-Retraction Theorem (Theorem 91). The rest of the proof is perfectly valid in $n$ dimensions. Thus, if we can extend the No-Retraction Theorem into $n$ dimensions, we will have successfully proven the $n$-dimensional Brouwer Fixed-Point Theorem.

4.1 $n$-Sphere Functions and Homotopy Classes

Definition 94 An $n$-sphere function is a continuous function $f : S^n \to S^n$.

Remark 95 We shall denote points on $B^{n+1}$ by $x = [\rho, \theta, \phi_1, \phi_2, \ldots, \phi_{n-1}]$, where $\rho$ is the distance from $x$ to the origin, $0 \leq \rho \leq 1$, and $\theta$ and each $\phi_i$ are angles such that $0 \leq \theta < 2\pi$ and $0 \leq \phi_i \leq \pi$ for all $i$ (see Figure 14). Note that we continue the convention of using square brackets to denote points in $B^{n+1}$. As with points in $S^1$, because $S^n$ is the subset of points in $B^{n+1}$ such that $\rho = 1$, the $\rho$ coordinate will be omitted, and points in $S^n$ will be written $y = (\theta, \phi_1, \ldots, \phi_{n-1})$, where $\theta$ and $\phi_i$ are defined as above. The base point $* \in S^n$ is defined to be $(0,0,\ldots,0)$. 
Figure 14: Spherical coordinates in $\mathbb{R}^3$.

**Theorem 96** Consider an $n$-sphere function $f : S^n \to S^n$. Then $f$ is homotopic to the constant function $* : S^n \to S^n$ defined by $*(x) = *$ for all $x \in S^n$, if and only if there exists a continuous extension $F : B^{n+1} \to S^n$ of $f$ onto the $(n+1)$-ball.

**Proof.** ($\Rightarrow$) Suppose $f : S^n \to S^n$ is homotopic to $*$. Then there exists a homotopy $G : S^n \times [0,1] \to S^n$ such that $G(\theta, \phi_1, \ldots, \phi_{n-1}, 0) = *$ and $G(\theta, \phi_1, \ldots, \phi_{n-1}, 1) = f(\theta, \phi_1, \ldots, \phi_{n-1})$. Now, define $F : B^{n+1} \to S^n$ by $F[\rho, \theta, \phi_1, \ldots, \phi_{n-1}] = G(\theta, \phi_1, \ldots, \phi_{n-1}, \rho)$. Note that when $\rho = 0$, $G(\theta, \phi_1, \ldots, \phi_{n-1}, \rho) = *$, so $F$ is well-defined at $\rho = 0$, and is thus defined as a function on $B^{n+1}$. Also, since $G$ is continuous, $F$ is continuous as well. Finally, $F[1, \theta, \phi_1, \ldots, \phi_{n-1}] = G(\theta, \phi_1, \ldots, \phi_{n-1}, 1) = f(\theta, \phi_1, \ldots, \phi_{n-1})$, which makes $F$ a continuous extension of $f$.

($\Leftarrow$) Suppose $f : S^n \to S^n$ extends to a continuous function $F : B^{n+1} \to S^n$. Define a function $G : S^n \times [0,1] \to S^n$ by $G(\theta, \phi_1, \ldots, \phi_{n-1}, t) = F[t, \theta, \phi_1, \ldots, \phi_{n-1}]$. Note that $F$ and $G$ are both continuous. By **Definition 78**, $G$ is a homotopy between the $n$-sphere functions $G(\theta, \phi_1, \ldots, \phi_{n-1}, 0)$ and $G(\theta, \phi_1, \ldots, \phi_{n-1}, 1)$. But $G(\theta, \phi_1, \ldots, \phi_{n-1}, 0) = F[0, \theta, \phi_1, \ldots, \phi_{n-1}]$, which is constant and thus homotopic to $*$, and $G(\theta, \phi_1, \ldots, \phi_{n-1}, 1) = F[1, \theta, \phi_1, \ldots, \phi_{n-1}] = f(\theta, \phi_1, \ldots, \phi_{n-1})$. Hence, $f \simeq *$. ■
In $S^1$, the homotopy classes of circle functions are represented by the functions $c_n(\theta) = n\theta$, where $n$ is the number of times the circle is “wrapped around itself.” The homotopy classes of $S^n$ can be represented in a similar manner. Consider the $n$-sphere function $c_m : S^n \to S^n$ defined by $c_m(\theta, \phi_1, \ldots, \phi_{n-1}) = (m\theta, \phi_1, \ldots, \phi_{n-1})$, $m \in \mathbb{Z}$. Notice that $c_m$ fixes all of the coordinates except $\theta$, which is multiplied by the integer $m$. In $S^2$, we can visualize $c_m$ as wrapping the area of $S^2$ around itself $m$ times about the axis connecting the sphere’s north and south poles. If a function is homotopic to $c_m$, we can call $m$ the function’s degree.

It should be noted that, unlike in $S^1$, the functions $c_0$ and $*$ are not equal in $S^n$. However, by constructing the homotopy $H : S^n \times [0, 1] \to S^n$ defined by

$$H(\theta, \phi_1, \ldots, \phi_{n-1}, t) = (0, (1-t)\phi_1, \ldots, (1-t)\phi_{n-1}),$$

we see that $H(\theta, \phi_1, \ldots, \phi_{n-1}, 0) = c_0$ and $H(\theta, \phi_1, \ldots, \phi_{n-1}, 1) = *$, which implies $c_0 \simeq *$.

As with $S^1$, all $n$-sphere functions have a unique degree. However, the proof of this goes beyond the scope of this paper, and will be submitted without proof. The reader may refer to [2] for more information.

**Theorem 97** Let $f : S^n \to S^n$ be an $n$-sphere function. Then there exists a unique $m \in \mathbb{Z}$ such that $f \simeq c_m$, where $c_m$ is the degree function described above.

### 4.2 No-Retraction Theorem in $n$ Dimensions

We are now prepared to prove the $n$-dimensional No retraction Theorem.

**Theorem 98** There exists no retraction from the $(n + 1)$-ball $B^{n+1}$ to its boundary $S^n$.

**Proof.** Suppose a retraction $r : B^{n+1} \to S^n$ exists. Then $r$ is continuous and $r[1, X] = X$ for all $X \in S^n$. Thus, on $S^n$, $r = id$, which has degree 1. But, Theorem 96 tells us that $r$ is homotopic to $*$ when restricted to $S^n$ since, by definition, it is a continuous extension of the identity function $id : S^n \to S^n$ onto $B^{n+1}$. As stated earlier, $* \simeq c_0$, so $r$ has degree 0 by transitivity. Theorem 97 tells us a function cannot have both degree 0 and degree 1. Thus, no such function $r$ can exist. ■

As was stated earlier, the $n$-dimensional case of the Brouwer Fixed-Point Theorem is a direct consequence of Theorem 98.
5 Fixed Points in Transformational Geometry

At this point in the paper, we leave the realm of topology and enter into the world of transformational geometry. While some familiarity with Euclidean geometry is preferred, statements of the relevant plane-geometric theorems can be found in Appendix B. Fixed points are an invaluable addition to the transformational geometer’s toolkit. In fact, isometries (distance-preserving transformations) can be completely classified by looking solely at the points they fix in the plane!

5.1 Isometries and the Three Points Theorem

Definition 99 A transformation of the plane is a function $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $\mathbb{R}^2$.

Definition 100 An isometry is a bijective transformation $\alpha$ of the plane that preserves distance. In other words, if $\alpha(A) = A'$ and $\alpha(B) = B'$, then $d(A, B) = d(A', B')$.

Theorem 101 Let $\alpha$ and $\beta$ be isometries. Then, without loss of generality, the composition $(\alpha \circ \beta) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(\alpha \circ \beta)(A) = \alpha(\beta(A))$ is an isometry.

Proof. Let $\alpha, \beta$ be isometries, and let $A, B \in \mathbb{R}^2$. Since $\beta$ is an isometry, for $\beta(A) = A'$ and $\beta(B) = B'$ we know $d(A, B) = d(A', B')$ by definition. Further, since $\alpha$ is an isometry, for $\alpha(A') = \alpha(\beta(A)) = A''$ and $\alpha(B') = \alpha(\beta(B)) = B''$ we know $d(A', B') = d(A'', B'')$. Hence, by transitivity, $d(A, B) = d(A'', B'')$, which proves that $(\alpha \circ \beta)$ is an isometry. ■

Now, based entirely on Definition 100, as well as on some elementary geometric theorems, we can prove two crucial fixed-point theorems, the second of which is a direct consequence of the first.

Theorem 102 Let $\alpha$ be an isometry, and let $\alpha$ fix two distinct points $A$ and $B$. Then $\alpha$ fixes the line $\ell$ which contains $A$ and $B$.

Proof. Let $\alpha$ be an isometry, and suppose for points $A, B \in \ell$, $A \neq B$, $\alpha(A) = A$ and $\alpha(B) = B$. Now, consider a point $C \in \ell$ distinct from $A$ and $B$. There are three distinct cases, two with identical proofs, which depend on the position of $C$ relative to $A$ and $B$.

Case 1 Suppose $C$ is between $A$ and $B$. Since $\alpha$ is an isometry, $d(A, C) = d(A, \alpha(C))$ and $d(B, C) = d(B, \alpha(C))$. Thus, $\alpha(C)$ must lie on the circle $A_C$ whose center is $A$ and whose
radius is $d(A, C)$, as well as the circle $B_C$ whose center is $B$ and whose radius is $d(B, C)$. Clearly, $A_C$ and $B_C$ intersect at $C$. But suppose the circles also intersect at another point $D$. If $D \in \ell$, then $D = C$. Thus, let $D \notin \ell$. Then there exists a triangle $\triangle ABD$ such that $AD + DB = AB$, which contradicts the Triangle Inequality (Theorem 142). Thus, $A_C \cap B_C = C$, which means that, to preserve distance, $\alpha(C) = C$ (see Figure 15).

![Figure 15: The circles $A_C$ and $B_C$ intersect only at the point $C$.](image)

**Case 2** Suppose $C$ is not between $A$ and $B$. Without loss of generality, assume $A$ is between $B$ and $C$. Again, $d(A, C) = d(A, \alpha(C))$ and $d(B, C) = d(B, \alpha(C))$, so $\alpha(C)$ must lie on circles $A_C$ and $B_C$ as defined in Case 1. These circles again intersect at $C$. Suppose they intersect at another point $D$. If $D \in \ell$, then $D = C$. Thus, suppose $D \notin \ell$. By Theorem 141 there is a triangle $\triangle ABD$ such that $BC = BD$ and $AC = AD$. Also, since $AB = AB$, it is clear by SSS (Theorem 143) that $\triangle ABD \cong \triangle ABC$. But, $A$, $B$, and $C$ are collinear, which means $D$ is on the same line as $A$, $B$, and $C$, which is precisely $\ell$. This contradicts our assumption that $D \notin \ell$. Hence, $A_C \cap B_C = C$, which means $\alpha(C) = C$ (see Figure 16). In all cases, $C$ is fixed by $\alpha$, and the proof is complete.

![Figure 16: The circles $A_C$ and $B_C$ intersect only at the point $C$.](image)
Theorem 103  Let $A$, $B$, and $C$ be non-collinear points, and let $\alpha$ be an isometry which fixes $A$, $B$, and $C$. Then $\alpha$ is the identity transformation $\text{id} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\text{id}(P) = P$ for all $P \in \mathbb{R}^2$.

**Proof.** Since $\alpha$ fixes $A$ and $B$, $\alpha$ also fixes $\overrightarrow{AB}$ by Theorem 102. Similarly, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are fixed by $\alpha$ as well. Since $A$, $B$, and $C$ are non-collinear, it is clear that $\triangle ABC$ is fixed and well-defined. Now, let $D$ be a point on $\triangle ABC$ which is distinct from points $A$, $B$, and $C$, and consider an arbitrary point $E \in \mathbb{R}^2$. If $E$ is on one of the three fixed lines, then $E$ is obviously fixed by $\alpha$. If $E$ is not on one of the three fixed lines, then the line $\overrightarrow{DE}$ intersects $\triangle ABC$ at $D$ and exactly one other point $F$ (see Figure 17). The points $D$ and $F$ are fixed by $\alpha$, and Theorem 102 tells us that $\overrightarrow{DF}=\overrightarrow{DE}$ is fixed by $\alpha$, so $\alpha(E) = E$. Because $E$ was arbitrary, $\alpha(P) = P$ for all $P \in \mathbb{R}^2$, making $\alpha$ the identity transformation. $\blacksquare$

![Figure 17: The line $\overrightarrow{DE}$ intersects $\triangle ABC$ in exactly two points.](image)

Restated, Theorem 103 says that if an isometry $\alpha$ fixes three non-collinear points, then it fixes the entire plane. We will now use Theorem 103 to prove another useful theorem: the Three Points Theorem.

**Theorem 104 (Three Points Theorem)** Let $\alpha, \beta$ be isometries. If $\alpha$ and $\beta$ agree on three non-collinear points $A$, $B$, and $C$, then $\alpha = \beta$.

**Proof.** Let $\alpha$ and $\beta$ be isometries and let $A$, $B$, and $C$ be non-collinear points such that $\alpha(A) = \beta(A)$, $\alpha(B) = \beta(B)$, and $\alpha(C) = \beta(C)$. If we apply $\alpha^{-1}$ to both sides of each equation, we get $A = \alpha^{-1}(\beta(A))$, $B = \alpha^{-1}(\beta(B))$, and $C = \alpha^{-1}(\beta(C))$. Since $(\alpha^{-1} \circ \beta)$ is an isometry by Theorem 101, we have that $(\alpha^{-1} \circ \beta)$ is an isometry which fixes three non-collinear points. By Theorem 103, $(\alpha^{-1} \circ \beta) = \text{id}$. Finally, applying $\alpha$ to both sides of this equation gives us $\alpha = \beta$, and the proof is complete. $\blacksquare$
5.2 Reflections and Rotations

**Definition 105** Let $A, B$ be distinct points. The midpoint of $AB$ is the point $M$ on $AB$ which is equidistant from points $A$ and $B$.

**Definition 106** Let $A, B$ be distinct points. The perpendicular bisector of $AB$ is the line perpendicular to $AB$ which intersects the midpoint $M$ of $AB$.

**Theorems 145** and **146** tell us that perpendicular bisectors and midpoints exist and are unique. Hence, we may define a transformation called a reflection as follows:

**Definition 107** Let $\ell$ be a line. The reflection in line $\ell$ is the transformation $\sigma_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\sigma_\ell$ fixes $\ell$ pointwise, and if $P \notin \ell$ and $P' = \sigma_\ell(P)$, then $\ell$ is the perpendicular bisector of $PP'$.

**Theorem 108** Reflections are isometries.

**Proof.** Let $\sigma_\ell$ be the reflection in arbitrary line $\ell$. To show it is an isometry, we must show that it is both injective and surjective, and that it preserves distance.

Thus, for some $P, Q \in \mathbb{R}^2$ and an arbitrary line $\ell$, let $\sigma_\ell(P) = \sigma_\ell(Q)$. Call this point $R$. Now, suppose $P \neq Q$. We will show that this is always contradictory, proving $\sigma_\ell$ is injective.

**Case 1** Suppose both $P$ and $Q$ are on $\ell$. Then the definition of $\sigma_\ell$ tells us that $\sigma_\ell(P) = P \neq Q = \sigma_\ell(Q)$, which contradicts our assumption that $\sigma_\ell(P) = \sigma_\ell(Q)$.

**Case 2** Suppose, without loss of generality, that $P \in \ell$ and $Q \notin \ell$. Then we know $\sigma_\ell(P) = P \in \ell$ and $\sigma_\ell(Q) = Q' \notin \ell$. It follows that $P = \sigma_\ell(P) = R = \sigma_\ell(Q) = Q'$, which implies that $R$ is both on and not on $\ell$. This is a contradiction.

**Case 3** Therefore, suppose $P, Q \notin \ell$. Since $\sigma_\ell(P) = \sigma_\ell(Q)$, $P$ and $Q$ are on the same side of $\ell$. Now, the definition of $\sigma_\ell$ says that $\ell$ is the perpendicular bisector of $PR$ and $QR$. Thus, points $P, Q$, and $R$ form $\triangle PQR$. Since $P$ and $Q$ are equidistant from $\ell$, it follows from **Theorem 148** that $\overrightarrow{PQ} \parallel \ell$, and thus $\angle QPR \cong \angle PQR$, which are both right angles (see Figure 18). This is a contradiction, since a triangle can only have one right angle.
Figure 18: If $P \neq Q$, then $\triangle PQR$ has two right angles.

To show that $\sigma_\ell$ is surjective, consider an arbitrary point $Q \in \mathbb{R}^2$. We show that there is a point $P$ such that $\sigma_\ell(P) = Q$. If $Q \in \ell$, then $\sigma_\ell(Q) = Q$ and the condition for surjectivity has been met. Thus, suppose $Q \notin \ell$. Construct the line $m$ such that $Q \in m$ and $\ell \perp m$. Let $R = \ell \cap m$. Now, construct the circle $R_Q$ whose center is $R$ and whose radius is $d(R,Q)$. The line $m$ intersects $R_Q$ at two points: $Q$, and another point $P$. It follows that $\ell$ is the perpendicular bisector of $PQ$. Hence, there exists a point $P$ such that $\sigma_\ell(P) = Q$.

Now we must prove that $d(P,Q) = d(P',Q')$ for all $P,Q \in \mathbb{R}^2$, where $P' = \sigma_\ell(P)$ and $Q' = \sigma_\ell(Q)$. As with the proof of the injectivity of $\sigma_\ell$, there are multiple cases based on where $P$ and $Q$ lie with respect to $\ell$. By proving that distance is preserved in all cases, we will have shown that $\sigma_\ell$ is an isometry. Cases 1, 2, and 3 of the following proof are taken with permission directly from [5, pp. 8-9]. Only some minor wording and notation has been changed, as has the approach to the conclusion of case 3. Case 4 is original.

**Case 1** Suppose that $P$ and $Q$ lie on $\ell$. Then $P' = P$ and $Q' = Q$, so $d(P',Q') = d(P,Q)$ as required.

**Case 2** Suppose, without loss of generality, that $P \in \ell$ and $Q \notin \ell$. Then $P = P'$ and $\ell$ is the perpendicular bisector of $\overline{QQ'}$. If $\overrightarrow{PQ} \perp \ell$, then $Q'$ lies on $\overrightarrow{PQ}$ and $P$ is the midpoint of $\overline{QQ'}$, so that $d(P,Q) = d(P,Q') = d(P',Q')$ as required. Otherwise, let $R$ be the point of intersection of $\ell$ with $\overline{QQ'}$. Since $\overrightarrow{PR} \cong \overrightarrow{PR}$, $\angle PRQ \cong \angle PRQ'$, and $\overrightarrow{QR} \cong \overrightarrow{QR}$, $\triangle PQR \cong \triangle PQ'R$ by **SAS** (**Theorem 149**) (see Figure 19). By **CPCTC** (**Theorem 144**) we have $d(P,Q) = d(P,Q') = d(P'Q')$ as required.
Figure 19: Triangles $\triangle PQR$ and $\triangle P'Q'R$ are congruent by SAS.

Case 3 Suppose that $P$ and $Q$ lie off of $\ell$ and on the same side of $\ell$. If $\overrightarrow{PQ} \perp \ell$, let $R$ be the point of intersection of $\ell$ with $\overrightarrow{PQ}$. If $d(P, R) > d(Q, R)$, then $d(P, Q) = d(P, R) - d(Q, R) = d(P', R) - d(Q', R) = d(P', Q')$, and similarly if $d(Q, R) > d(P, R)$. If $\overrightarrow{PQ} \not\perp \ell$, then let $R$ be the point of intersection of $\ell$ with $\overrightarrow{PP'}$ and let $S$ be the point of intersection of $\ell$ with $\overrightarrow{QQ'}$. Then $\overrightarrow{PR} \cong \overrightarrow{P'R}$, $\angle PRS \cong \angle P'RS$, and $\overrightarrow{RS} \cong \overrightarrow{RS}$, so $\triangle PRS \cong \triangle P'RS$ by SAS so that $\overrightarrow{PS} \cong \overrightarrow{P'S}$ by CPCTC. Now, $\angle RSQ \cong \angle RSQ'$ and $\angle RSP \cong \angle RSP'$, so $\angle PSQ \cong \angle P'SQ'$, and $\overrightarrow{SQ} \cong \overrightarrow{SQ'}$, so $\triangle PSQ \cong \triangle P'SQ'$ by SAS (see Figure 20). It follows that $d(P, Q) = d(P'Q')$ by CPCTC.

Figure 20: Triangles $\triangle PSQ$ and $\triangle P'SQ'$ are congruent by SAS.
Case 4 Suppose that both $P$ and $Q$ lie off $\ell$ and on opposite sides of $\ell$. If $\overrightarrow{PQ} \perp \ell$, let $R$ be the point of intersection of $\ell$ with $\overrightarrow{PQ}$. Since $P, P', Q,$ and $Q'$ are collinear, $d(P, Q) = d(P, R) + d(R, Q) = d(P', R) + d(R, Q') = d(P', Q')$ as desired. If $\overrightarrow{PQ} \not\perp \ell$, then let $R$ be the point of intersection of $\ell$ with $\overrightarrow{PP'}$, let $S$ be the intersection of $\ell$ with $\overrightarrow{QQ'}$, and let $T$ be the point of intersection of $\ell$ with $\overrightarrow{PQ}$. Then $\overrightarrow{PR} \cong \overrightarrow{P'R}$, $\angle PRT \cong \angle P'R'T$, and $\overrightarrow{RT} \cong \overrightarrow{RT'}$, so $\triangle PRT \cong \triangle P'R'T$ by SAS. This implies that $d(P, T) = d(P', T)$ by CPCTC.

Similarly, $\overrightarrow{QS} \cong \overrightarrow{Q'S}$, $\angle QST \cong \angle Q'ST$, and $\overrightarrow{ST} \cong \overrightarrow{ST'}$, so $\triangle QST \cong \triangle Q'ST$ by SAS, which implies that $d(Q, T) = d(Q', T)$. Since vertical angles are congruent, $\mu(\angle P'TR) = \mu(\angle Q'TS)$, $\mu(\angle RTP) = \mu(\angle STQ)$, and $\mu(\angle PTQ') = \mu(\angle QTP')$ (see Figure 21). Hence, $2\mu(\angle P'TR) + 2\mu(\angle RTP) + 2\mu(\angle PTQ') = 2\pi$, or $\mu(\angle P'TR) + \mu(\angle RTP) + \mu(\angle PTQ') = \pi$.

Thus, $P', T$, and $Q'$ are collinear, as are $P, T$, and $Q$ by hypothesis, and it follows that $d(P, Q) = d(P, T) + d(T, Q) = d(P', T) + d(T, Q') = d(P', Q')$ as desired.

Hence, we have shown that in every case, $\sigma_\ell$ satisfies all the requirements of an isometry. \hfill\blacksquare

![Figure 21: Vertical angles are congruent, making points $P'$, $T$, and $Q'$ collinear.](image)

Remark 109 Angles are measured counterclockwise in degrees. Every degree measure is equivalent to some real number in the interval $(-180, 180]$. When expressing angle measure, $\Theta$ will be taken to be any real number, while $\Theta^\circ$ will be taken to be a real number whose angle measure in degrees is equivalent to a real number in the interval $(-180, 180]$. 

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Definition 110 Let $C$ be a point and let $\Theta \in \mathbb{R}$. The rotation about $C$ of $\Theta^\circ$ is the transformation $\rho_{C,\Theta} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

1. $\rho_{C,\Theta}(C) = C$.

2. If $P \neq C$ and $P' = \rho_{C,\Theta}(P)$, then $CP' = CP$ and $\mu(\angle PCP') = \Theta^\circ$.

Theorem 111 Rotations are isometries.

Proof. To show that $\rho_{C,\Theta}$ is an isometry, we must show that it is injective and surjective, and that it preserves distance.

Consider the rotation $\rho_{C,-\Theta}$. It is intuitively clear that for a point $P \neq C$, $\rho_{C,\Theta}(\rho_{C,-\Theta}(P))$ rotates a point $P$ $\Theta^\circ$ clockwise and then $\Theta^\circ$ counterclockwise, thus bringing $P$ back to its original position. Likewise, $\rho_{C,-\Theta}(\rho_{C,\Theta}(P))$ rotates a point $P$ $\Theta^\circ$ counterclockwise and then $\Theta^\circ$ clockwise, again bringing $P$ back to its original position. Finally, since $\rho_{C,\Theta}$ and $\rho_{C,-\Theta}$ both fix $C$, both aforementioned compositions fix $C$ as well. Thus,

$$\rho_{C,\Theta}(\rho_{C,-\Theta}(P)) = \rho_{C,-\Theta}(\rho_{C,\Theta}(P)) = P$$

for all $P \in \mathbb{R}^2$, so the two rotations are inverses of one another. Since every $\rho_{C,\Theta}$ is invertible, it follows from Theorem 57 that $\rho_{C,\Theta}$ is bijective.

The following part of the proof is again taken with permission from [5, pp. 27]. Details and a figure have been added to aid the reader, but the proof itself remains unchanged.
Let $\rho_{C,\Theta}$ be a rotation. Let $C$, $P$, and $Q$ be points with $P$ and $Q$ distinct; let $P' = \rho_{C,\Theta}(P)$ and $Q' = \rho_{C,\Theta}(Q)$. If $P = C$, then by definition, $PQ = CQ = CQ' = P'Q'$. So assume that $C$, $P$, and $Q$ are distinct.

If $C$, $P$, and $Q$ are non-collinear, then $PC \cong P'C$, $\mu(\angle PCQ) = \mu(\angle PCQ') + \mu(\angle Q'CQ) = \Theta + \mu(\angle PCQ') = \mu(\angle P'CQ) + \mu(\angle PCQ') = \mu(\angle P'CQ')$, and $CQ \cong CQ'$, which implies that $\triangle PCQ \cong \triangle P'CQ'$ by SAS, and $d(P, Q) = d(P', Q')$ since CPCTC (see Figure 23).

![Figure 23: Since $\triangle PCQ \cong \triangle P'CQ'$, $d(P, Q) = d(P', Q')$.](image)

If $C$, $P$, and $Q$ are collinear with $P$ between $C$ and $Q$, then $d(P, Q) = d(C, Q) - d(C, P) = d(C, Q') - d(C, P') = d(P', Q')$, since $d(C, P) = d(C, P')$ and $d(C, Q) = d(C, Q')$ by definition, and similarly for $Q$ between $C$ and $P$. But if $C$ is between $P$ and $Q$, then $C$ is also between $P'$ and $Q'$, since $\mu(\angle PCP') = \mu(\angle QCQ') = \Theta$. Therefore, $d(P, Q) = d(C, P) + d(C, Q) = d(C, P') + d(C, Q') = d(P', Q')$. \[\blacksquare\]

In order to prove that two isometries are equal, we must show that they agree at every point in $\mathbb{R}^2$. Unfortunately, there are uncountably many points in $\mathbb{R}^2$, so a point-by-point proof is impossible at best. However, armed with the Three Points Theorem, we need only verify that the two isometries agree at three non-collinear points. This is exactly the approach we will take to conclude the section with our next result.

55
**Theorem 112** A non-identity isometry is the composition of two reflections in distinct intersecting lines if and only if it is a rotation.

**Proof.** To prove the theorem, we need only show that \( \sigma_m \circ \sigma_\ell \) and \( \rho_{C,2\Theta} \) agree on three non-collinear points. Then the Three Points Theorem (Theorem 104) tells us that the two isometries are equal. Throughout the proof, the reader may refer to Figure 24.

> Figure 24: The rotation \( \rho_{C,2\Theta} \) is equivalent to the composition of \( \sigma_\ell \) and \( \sigma_m \).

\((\to)\) Let \( \ell, m \) be distinct lines which intersect at point \( C \), and let \( \Theta \) be the measure of an angle from \( \ell \) to \( m \). Now, consider the point \( C \). Observe that

\[
(\sigma_m \circ \sigma_\ell)(C) = \sigma_m(\sigma_\ell(C)) = \sigma_m(C) = C,
\]

and,

\[
\rho_{C,2\Theta}(C) = C.
\]

Thus, \( \sigma_m \circ \sigma_\ell \) and \( \rho_{C,2\Theta} \) agree at \( C \).

Next, let \( L \) be a point on \( \ell \) distinct from \( C \). Consider the circle \( C_L \) whose center is \( C \) and whose radius is \( d(C, L) \). Let \( M \) be the point at which \( C_L \) and \( m \) intersect such that \( \mu(\angle MCL) = \Theta \), and let \( L' = \rho_{C,2\Theta}(L) \). Consider the segment \( LL' \). If \( \Theta = \pm 90^\circ \), then \( C \) is the midpoint of \( LL' \) and so \( m \) is the perpendicular bisector of \( LL' \). If \( \Theta \neq \pm 90^\circ \), let \( N \) be the point at which \( LL' \) intersects \( m \). Then \( NC \cong NC', \mu(\angle LCN) = \Theta = 2\Theta - \Theta = \mu(\angle L'CL) - \mu(\angle LCN) = \mu(\angle L'CN) \), and \( LC \cong LC' \), so \( \triangle NCL \cong \triangle NCL' \) by SAS. Thus, \( LN \cong NL' \).
and \( \angle LNC \cong \angle CNL' \). But \( L, N, \) and \( L' \) are collinear, so \( \mu(\angle LNC) + \mu(\angle CNL') = 180^\circ \), which implies \( \mu(\angle LNC) = \mu(\angle CNL') = 90^\circ \). Hence, \( m \) is the perpendicular bisector of \( LL' \). It follows that \( \sigma_m(L) = L' \). Consequently,

\[
(\sigma_m \circ \sigma_\ell)(L) = \sigma_m(\sigma_\ell(L)) = \sigma_m(L) = L' = \rho_{C,2\Theta}(L).
\]

Therefore, \( \sigma_m \circ \sigma_\ell \) and \( \rho_{C,2\Theta} \) also agree at \( L \).

Finally, let \( J = \sigma_\ell(M) \). By the definition of \( \sigma_\ell, \ell \) is the perpendicular bisector of \( JM \).

Further, since \( \ell \) intersects the vertex \( C \) of \( \Delta MCJ \), Theorem 150 tells us that \( \Delta MCJ \) is isosceles, and that \( \overline{MC} \cong \overline{CJ} \). It follows that \( J \) lies on \( C_L \), and Theorem 151 tells us that \( \ell \) bisects \( \angle MCJ \). But \( \mu(\angle LCJ) = \Theta \), so \( \mu(\angle MCJ) = 2\Theta \). Hence, \( M = \rho_{C,2\Theta}(J) \).

Consequently,

\[
(\sigma_m \circ \sigma_\ell)(J) = \sigma_m(\sigma_\ell(J)) = \sigma_m(M) = M = \rho_{C,2\Theta}(J),
\]

which shows that \( \sigma_m \circ \sigma_\ell \) and \( \rho_{C,2\Theta} \) agree at \( J \).

The two transformations agree at non-collinear points \( C, L, \) and \( J \) as described above. Hence, by the Three Points Theorem, \( \sigma_m \circ \sigma_\ell = \rho_{C,2\Theta} \).

\( \leftarrow \) Consider the rotation \( \rho_{C,2\Theta} \). Let \( \ell \) be any line through \( C \) and let \( m \) be the unique line through \( C \) such that the measure of an angle from \( \ell \) to \( m \) is \( \Theta \) (see Figure 25). By our previous implication, \( \rho_{C,2\Theta} = \sigma_m \circ \sigma_\ell \).

---

**Figure 25**: Lines \( \ell \) and \( m \) go through \( C \), and the measure of an angle from \( \ell \) to \( m \) is \( \Theta \).
6 Conclusion

In this paper, we proved the general Brouwer Fixed-Point Theorem using topology, and we proved and applied the Three Points Theorem in transformational plane geometry. We have shown that the one-dimensional case of the Brouwer Fixed-Point Theorem is a direct consequence of the Intermediate Value Theorem, and that the case in \( n \) dimensions follows immediately from the \( n \)-Dimensional No-Retraction Theorem. We have shown that the Three Points Theorem is a consequence of the fact that an isometry which fixes three noncollinear points fixes every point. Finally, we applied the Three Points Theorem to demonstrate the relationship between reflections and rotations in the plane.
A Proof of Theorem 85

Recall that Theorem 85 states that for every circle function $f$ there exists a unique $n \in \mathbb{Z}$ such that $f$ is homotopic to $c_n(\theta) = n\theta$. Thus, we will need to prove the existence of $n$ as well as the uniqueness of $n$. The former is realized in Theorem 138, and the latter in Theorem 140. Together, these two theorems produce the desired result.

Up to this point, we have only dealt with two of the “three C’s” of topology, i.e., continuity and connectedness. In this appendix, we will explore the “third C:” compactness.

A.1 Compactness

Definition 113 Let $X$ be a topological space, $A \subseteq X$, and let $\mathcal{O}$ be a collection of subsets of $X$. If $A$ is contained in the union of the sets of $\mathcal{O}$, then $\mathcal{O}$ is called a cover of $A$. Further, if all the sets in $\mathcal{O}$ are open, we call $\mathcal{O}$ an open cover of $A$ (see Figure 26). Finally, if $\mathcal{O}'$ is a subcollection of $\mathcal{O}$ that is also a cover of $A$, then $\mathcal{O}'$ is called a subcover of $\mathcal{O}$.

![Figure 26: An open cover of $A$.](image)

Definition 114 Let $X$ be a topological space, and let $A \subseteq X$. Then the topological space $X$ is compact if every open cover of $X$ has a finite subcover. Further, the subset $A$ is compact in $X$ if $A$ is compact in the subspace topology inherited from $X$. 

Lemma 115 Let $X$ be a topological space and let $A \subseteq X$. Then $A$ is compact in $X$ if and only if every cover of $A$ by sets open in $X$ has a finite subcover.

Proof. $(\rightarrow)$ Let $A$ be compact in $X$. Suppose $\mathcal{O}$ is a cover of $A$ by open sets in $X$. Then $\mathcal{O}' = \{O \cap A \mid O \in \mathcal{O}\}$ is a cover of $A$ by open sets in $A$. Since $A$ is compact in $X$, there exists a finite subcover $\{O_1 \cap A, O_2 \cap A, \ldots, O_n \cap A\}$ of $\mathcal{O}'$. But then $\{O_1, O_2, \ldots, O_n\}$ is a finite subcover of $\mathcal{O}$. Thus, every cover of $A$ by sets open in $X$ has a finite subcover.

$(\leftarrow)$ Suppose every cover of $A$ by sets open in $X$ has a finite subcover. Let $\mathcal{O} = \{O_\beta\}_{\beta \in B}$ be an arbitrary cover of $A$ by open sets in $A$. By the definition of the subspace topology (Definition 24), for each $O_\beta$ there is an open set $U_\beta \in X$ such that $O_\beta = U_\beta \cap A$. Since $\mathcal{O}$ is a cover of $A$ by open sets in $A$, $\mathcal{O}' = \{U_\beta\}_{\beta \in B}$ is a cover of $A$ by open sets in $X$. By hypothesis, $\mathcal{O}'$ has a finite subcover $\{U_1, U_2, \ldots, U_n\}$. Consequently, $\mathcal{O}$ has a finite subcover $\{O_1, O_2, \ldots, O_n\}$. It follows that every cover of $A$ by open sets in $A$ has a finite subcover, and therefore $A$ is compact in $X$.

Theorem 116 Let $X$ be a topological space and let $D$ be compact in $X$. If $C$ is closed in $X$ and $C \subseteq D$, then $C$ is compact in $X$.

Proof. Let $C$ be closed in $X$ and let $C \subseteq D$. Since $C$ is closed, $C^c$ is open. Now, let $\mathcal{O}$ be an open cover of $C$. Then $\mathcal{O}' = \mathcal{O} \cup \{C^c\}$ is an open cover of $X$, and thus is an open cover of $D$. Since $D$ is compact, there is a finite subcollection of $\mathcal{O}'$ that covers $D$. Consequently, $C$ is covered by the sets in the finite subcover of $\mathcal{O}'$ that were originally in $\mathcal{O}$. Therefore, there is a finite subcollection of $\mathcal{O}$ that covers $C$, implying that $C$ is compact in $X$.

Theorem 117 Let $X, Y$ be topological spaces, and let $f : X \to Y$ be continuous. If $A$ is compact in $X$, then $f(A)$ is compact in $Y$.

Proof. Let $f : X \to Y$ be continuous and let $A \subseteq X$ be compact. Now, let $\mathcal{O}$ be a cover of $f(A)$ by open sets in $Y$. Since $f$ is continuous, $f^{-1}(O)$ is open in $X$ for every set $O \in \mathcal{O}$. Thus, $\mathcal{O}' = \{f^{-1}(O) \mid O \in \mathcal{O}\}$ is a cover of $A$ by open sets in $X$. Since $A$ is compact in $X$, there exists a finite subcollection $\{f^{-1}(O_1), f^{-1}(O_2), \ldots, f^{-1}(O_n)\}$ of sets in $\mathcal{O}'$ that covers $A$. Consequently, the finite collection of open sets $\{O_1, O_2, \ldots, O_n\}$ covers $f(A)$. Hence, a finite subcover of $\mathcal{O}$ exists, which implies $f(A)$ is compact.
A.2 Compact Sets in \( \mathbb{R} \); Hausdorff Spaces

Lemma 118 (Nested Intervals Lemma) Let \( \{[a_n, b_n]\}_{n \in \mathbb{N}} \) be a collection of nonempty closed bounded intervals in \( \mathbb{R} \) such that \([a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]\) for all \( n \in \mathbb{N} \). Then \( \bigcap_{n=1}^{\infty} [a_n, b_n] \) is nonempty.

Proof. Assume that \([a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]\) for every \( n \in \mathbb{N} \). It follows that the interval endpoints satisfy the following inequalities:

\[
a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.
\]

Now, the set \( \{a_n\}_{n \in \mathbb{N}} \) is bounded above by every \( b_n \) and has a least upper bound \( A \) by Axiom 10. Similarly, \( \{b_n\}_{n \in \mathbb{N}} \) is bounded below by each \( a_n \) and has a greatest lower bound \( B \). Notice that \( A \leq B \), so \( [A, B] \) is nonempty.

Let \( x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \). Then \( x \in [a_n, b_n] \) for all \( n \), which implies that \( a_n \leq x \leq b_n \) for all \( n \). Consequently, \( A \leq x \leq B \), and it follows that \( x \in [A, B] \). Hence, \( \bigcap_{n=1}^{\infty} [a_n, b_n] \subseteq [A, B] \).

Now, let \( x \in [A, B] \). Then \( A \leq x \leq B \), which tells us \( a_n \leq x \leq b_n \) for all \( n \in \mathbb{N} \). Thus, \( x \) is an element of every interval \([a_n, b_n]\). By the definition of intersection, \( x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \). Hence, \( [A, B] \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n] \).

Since each is a subset of the other, we have proven that \( \bigcap_{n=1}^{\infty} [a_n, b_n] = [A, B] \), which we stated was nonempty. ■

Theorem 119 Let \( \mathbb{R} \) have the standard topology. Then every closed and bounded interval \([a, b]\) is compact in \( \mathbb{R} \).

Proof. Let \( \mathcal{O} \) be a cover of \([a, b]\) with open sets in \( \mathbb{R} \). Now, suppose \([a, b]\) is not compact. Then there exists no finite subcover of \( \mathcal{O} \) covering \([a, b]\). Consider the intervals \([a, \frac{a+b}{2}]\) and \([\frac{a+b}{2}, b]\) obtained by dividing \([a, b]\) in half. The collection \( \mathcal{O} \) covers both intervals. At least one of the two has no finite subcollection of \( \mathcal{O} \) that covers it. Without loss of generality, we will call this interval \([a_1, b_1]\). Similarly, we can choose a half of \([a_1, b_1]\) which can not be covered by finitely many sets of \( \mathcal{O} \), and we will call it \([a_2, b_2]\). We repeat the process indefinitely, so that given an interval \([a_n, b_n]\) which is not covered by a finite subcollection of \( \mathcal{O} \), we halve it, choose an arbitrary half not covered finitely by \( \mathcal{O} \), and denote it by \([a_{n+1}, b_{n+1}]\).
Now, consider the collection of intervals \( \{[a_n, b_n]\}_{n \in \mathbb{N}} \). For each \( n \in \mathbb{N} \) we know that 
\[ [a_{n+1}, b_{n+1}] \subset [a_n, b_n], \quad b_n - a_n = \frac{b-a}{2^n}, \] and \([a_n, b_n]\) is not covered by a finite subcollection of \( \mathcal{O} \). By the Nested Intervals Lemma (Lemma 118), it follows that \( \bigcap_{n=1}^{\infty} [a_n, b_n] \) is nonempty. Let \( x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \). Consequently, \( x \in [a, b] \), so there exists an \( O \in \mathcal{O} \) such that \( x \in O \). Since \( U \) is open in \( \mathbb{R} \), there exists an \( \epsilon > 0 \) such that \( \overline{B}(x, \epsilon) = (x - \epsilon, x + \epsilon) \subseteq U \). Let \( N \in \mathbb{N} \) be large enough that \( \frac{b-a}{2^N} < \epsilon \). Since \( x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \), it follows that \( x \in [a_N, b_N] \). Further, since \( b_N - a_N = \frac{b-a}{2^N} \), it follows that \( [a_N, b_N] \subset (x - \epsilon, x + \epsilon) \subseteq U \). This tells us that \([a_N, b_N]\) is covered by a single element of \( \mathcal{O} \), which contradicts the fact that \([a_N, b_N]\) is not covered by a finite subcollection of \( \mathcal{O} \). Hence, there exists a finite subcollection of \( \mathcal{O} \) that covers \([a, b]\), and thus \([a, b]\) is compact.

To prove the Heine-Borel Theorem on \( \mathbb{R} \), which says that a set in \( \mathbb{R} \) is compact if and only if it is closed and bounded, we will need to briefly introduce a new type of topological space called a Hausdorff space. We will then prove that compact sets in Hausdorff spaces are closed, at which point we will have the tools to prove the Heine-Borel Theorem on \( \mathbb{R} \).

**Definition 120** A topological space \( X \) is called a Hausdorff space, or just Hausdorff, if for every distinct \( x, y \in X \) there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

![Figure 27: In a Hausdorff space, distinct points have distinct neighborhoods.](image)

**Lemma 121** The real line \( \mathbb{R} \) is Hausdorff.

**Proof.** Let \( x, y \in \mathbb{R} \) be distinct. Without loss of generality, assume \( x < y \). Then \( x \in (x - 1, \frac{x+y}{2}) \) and \( y \in (\frac{x+y}{2}, y + 1) \). Since \( x < y \), it follows that the intervals are disjoint. Hence, \( \mathbb{R} \) is Hausdorff. ■
Theorem 122 Let $X$ be a Hausdorff space and $A$ be compact in $X$. Then $A$ is closed in $X$.

Proof. Since $X$ is Hausdorff, we know that for every $a \in A$ and every $x \in A^c$, there exist disjoint open sets $U_a$ and $V_a$ such that $x \in U_a$ and $a \in V_a$. Thus, $\mathcal{O} = \{V_a\}_{a \in A}$ is an open cover of $A$. Since $A$ is compact, there exists a finite subcover $\{V_{a_1}, V_{a_2}, \ldots, V_{a_n}\}$ of $\mathcal{O}$. Now, let $V = \bigcup_{i=1}^{n} V_{a_i}$ and let $U_x = \bigcap_{i=1}^{n} U_{a_i}$. Then $U$ and $V$ are open, $A \subseteq V$, and $x \in U_x$. Further, since $U_{a_i}$ and $V_{a_i}$ are disjoint for each $i$, we know $U_x$ and $V$ are also disjoint. It follows that $U_x$ and $A$ are disjoint. Hence, for every $x \in A^c$, there exists an open set $U_x$ such that $x \in U_x \subseteq A^c$. By the Union Lemma (Lemma 4), $\bigcup_{x \in A^c} U_x = A^c$, which is open by Definition 1: A4. Consequently, $A$ is closed by definition. ■

Theorem 123 (Heine-Borel Theorem on $\mathbb{R}$) Let $A \subseteq \mathbb{R}$, where $\mathbb{R}$ has the standard topology. Then $A$ is compact if and only if it is closed and bounded.

Proof. ($\Rightarrow$) Let $A \subseteq \mathbb{R}$ be compact. Since $\mathbb{R}$ is Hausdorff, it is closed by Theorem 122. Now, consider the collection of intervals $\mathcal{O} = \{((-n, n))_{n \in \mathbb{N}}\}$ centered at 0. The collection $\mathcal{O}$ is an open cover of $A$. Since $A$ is compact, a finite number of intervals in $\mathcal{O}$ cover $A$. Thus there exists $N \in \mathbb{N}$ such that $A \subseteq [-N, N]$. Hence, $A$ is bounded below by $-N$ and above by $N$.

($\Leftarrow$) Suppose $A$ is closed and bounded. Let $M$ be the least upper bound of $A$ and let $m$ be the greatest lower bound of $A$. Then $A \subseteq [m, M]$. It follows from Theorem 116 that $A$ is compact. ■

A.3 Extreme Value Theorem; Special Case of the Lebesgue Number Lemma

Lemma 124 Let $A$ be a compact subset of $\mathbb{R}$. Then there exists $m, M \in A$ such that $m \leq a \leq M$ for all $a \in A$.

Proof. We will proceed with the proof of the existence of $M$. The set $A$ is compact in $\mathbb{R}$; thus, it is closed and bounded by Theorem 123. Since $A$ is bounded in $\mathbb{R}$, $A$ is bounded from above. Axiom 10 tells us that $A$ has a least upper bound $M$. By definition, $a \leq M$ for all $a \in A$.

Now, suppose $M \notin A$. Since $A$ is closed, there is a neighborhood $(M - \epsilon, M + \epsilon)$ which
contains no points in \( A \). Thus, \( A \) is bounded above by every point in \((M - \epsilon, m + \epsilon)\). Thus, \( M - \frac{\epsilon}{2} \) is an upper bound of \( A \) which is less than \( M \). This contradicts the fact that \( M \) is the least upper bound of \( A \). Hence, \( M \in A \), and the proof is complete.

The proof of the existence of \( m \) is similar. ■

**Theorem 125 (Extreme Value Theorem)** Let \( X \) be compact and let \( f : X \to \mathbb{R} \) be continuous. Then there exist points \( a, b \in X \) such that \( f(a) \leq f(x) \leq f(b) \) for all \( x \in X \).

**Proof.** Since \( f \) is continuous and \( X \) is compact, \( f(X) \) is a compact subset of \( \mathbb{R} \). Therefore, by **Lemma 124**, \( f(X) \) contains a maximum value \( M \) and a minimum value \( m \). Since \( m, M \in f(X) \), there exist points \( a, b \in X \) such that \( f(a) = m \) and \( f(b) = M \). Now, for all \( x \in X \), \( f(x) \in f(X) \) by definition. Hence, \( f(a) = m \leq f(x) \leq M = f(b) \), and the proof is complete. ■

Restated, the Extreme Value Theorem says that if a set \( X \) is compact and a function \( f \) is continuous, then \( f(X) \) contains a minimum and a maximum value. The next theorem is a special case of the more general Lebesgue Number Lemma. It is a vital tool whose corollary we will use in the proof of **Theorem 137**, which states that every circle function has a function called a **lifting** associated with it. First, we need a brief definition and lemma to make our notation less cumbersome.

**Definition 126** Let \( A \subseteq \mathbb{R} \). For a point \( x \in \mathbb{R} \), the **distance from \( x \) to \( A \)** is given by the function \( f_A : \mathbb{R} \to \mathbb{R} \) defined by

\[
f_A(x) = \text{glb} \left\{|x - a| \mid a \in A\right\},
\]

where \( \text{glb} \) stands for greatest lower bound.

**Lemma 127** The function \( f_A : \mathbb{R} \to \mathbb{R} \) is continuous.

**Proof.** Let \( x, y \in \mathbb{R} \) and let \( A \subseteq \mathbb{R} \). By the Triangle Inequality, \( |x - a| \leq |x - y| + |y - a| \) for all \( a \in A \). Taking the \( \text{glb} \) of both sides with respect to \( a \) yields \( \text{glb}|x - a| \leq \text{glb}|x - y| - \text{glb}|y - a| \). But \( \text{glb}|x - a| \) and \( \text{glb}|y - a| \) are precisely \( f_A(x) \) and \( f_A(y) \) respectively. Further, since \( |x - y| \) is constant with respect to \( a \), \( \text{glb}|x - y| = |x - y| \). Thus, \( f_A(x) \leq |x - y| + f_A(y) \), and \( f_A(x) - f_A(y) \leq |x - y| \). Similarly, \( f_A(y) - f_A(x) \leq |x - y| \), so \( |f_A(x) - f_A(y)| \leq |x - y| \).
Now, let \( x_0 \in \mathbb{R} \) and \( \epsilon > 0 \). Choose \( \delta = \epsilon \). By our previous inequalities, \( \delta > |x - x_0| \) implies \( \epsilon > |x - x_0| \geq |f_A(x) - f_A(x_0)| \). Therefore, \( f_A \) is continuous at \( x_0 \) by Definition 41.

Since \( x_0 \) was arbitrary, the function \( f_A \) is continuous over \( \mathbb{R} \).

\[ \text{Theorem 128 (Lebesgue Number Lemma on } \mathbb{R} \text{)} \]

Let \( O \) be a cover of the closed bounded interval \([a, b]\) by sets that are open in \( \mathbb{R} \). Then there exists a Lebesgue number \( \lambda > 0 \) such that for every \( x \in [a, b] \), there exists an \( O \in \mathcal{O} \) such that \((x - \lambda, x + \lambda) \subseteq O \).

\[ \textbf{Proof.} \] Let \( O \) be a cover of \([a, b]\) by sets open in \( \mathbb{R} \). Suppose for some set \( O \in \mathcal{O} \), \([a, b] \subseteq O \). Then the interval \([a, b]\) is already a subdivision which satisfies the result. Thus, suppose \([a, b]\) is not a subset of any \( O \in \mathcal{O} \).

Since \([a, b]\) is compact, there is a finite subcollection \( \{O_1, O_2, \ldots, O_n\} \) of sets in \( \mathcal{O} \) that covers \([a, b]\). For each \( i = 1, 2, \ldots, n \), let \( C_i = [a, b] - O_i \). Since we assumed \([a, b]\) is not a subset of any \( O_i \), we know every \( C_i \) is nonempty. Now, define \( f: [a, b] \to \mathbb{R} \) by

\[
f(x) = \frac{1}{n} \sum_{i=1}^{n} f_{C_i}(x).
\]

The function \( f(x) \) is the average distance from \( x \) to each set \( C_i \). Note that for all \( x \), each \( f_{C_i}(x) \geq 0 \) by its definition. Thus, \( f(x) \geq 0 \).

Now, consider an arbitrary point \( x \in [a, b] \). Since the sets \( O_i \) cover \([a, b]\), \( x \in O_k \) for some \( k = 1, 2, \ldots, n \). \( O_k \) is open in \( \mathbb{R} \), so it is a union of open balls (intervals). Consider an arbitrary open ball \( B \) such that \( x \in B \). By Lemma 74, there exists an \( \epsilon > 0 \) such that \( B(x, \epsilon) = (x - \epsilon, x + \epsilon) \subseteq B \subseteq O_k \). It follows that \( f_{C_k}(x) > \epsilon \), and therefore \( f(x) > \frac{\epsilon}{n} > 0 \).

Since each \( f_{C_i} \) is continuous by Lemma 127, and the sum of continuous functions is continuous by Theorem 46, \( f \) is continuous. Further, since \([a, b]\) is compact, the Extreme Value Theorem (Theorem 125) tells us that \( f([a, b]) \) has a minimum value \( \lambda \). Since we showed earlier that \( f(x) > 0 \) for all \( x \in [a, b] \), it follows that \( \lambda > 0 \).

Finally, we will show that \( \lambda \) is the desired Lebesgue number. Thus, let \( x \in [a, b] \) be arbitrary, and consider the interval \((x - \lambda, x + \lambda)\). Suppose \((x - \lambda, x + \lambda) \not\subseteq O_i \) for all
Then for all $i$, $f_{C_i}(x) < \lambda$, which implies that
\[
f(x) = \frac{f_{C_1}(x) + f_{C_2}(x) + \ldots + f_{C_n}(x)}{n} < \frac{\lambda + \lambda + \ldots + \lambda}{n} = \frac{n\lambda}{n} = \lambda.
\]
This, however, directly contradicts the fact that $\lambda$ is the minimum value of $f$ over $[a, b]$. Consequently, there is an $O_i$ such that $(x - \lambda, x + \lambda) \subseteq O_i$, and $\lambda$ is the Lebesgue number we sought.

Corollary 129 Let $O$ be a cover of the interval $[a, b]$ by sets open in $\mathbb{R}$. Then there is a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ of $[a, b]$ such that for all $j = 1, 2, \ldots, n$ there exists an $O_j \in O$ such that $[a_{j-1}, a_j] \subseteq O_j$.

Proof. By Theorem 128, there exists a $\lambda > 0$ such that for every $x \in [a, b]$ every interval $(x - \lambda, x + \lambda)$ is a subset of some $O_i \in O$. Thus, let $a = a_0 < a_1 < \ldots < a_n = b$ be a subdivision such that $a_i - a_{i-1} < 2\lambda$ for all $i = 1, 2, \ldots, n$, and let $x_i = \frac{a_{i-1} + a_i}{2}$. It follows that $[a_{i-1}, a_i] \subseteq (x_i - \lambda, x_i + \lambda) \subseteq O_i$ for some $O_i \in O$.

A.4 Circle Functions Revisited; Liftings

At this point in the thesis, we will refer to points specifically in $S^1$ by their complex exponential format. Thus, the point in $S^1$ previously denoted $\theta$ will be expressed as $e^{i\theta} = \cos \theta + i \sin \theta$. Notice that the function $c_n : S^1 \rightarrow S^1$, which was previously defined by $c_n(\theta) = n\theta$, is now redefined by the function $c_n(e^{i\theta}) = e^{in\theta}$.

Remark 130 Every circle function $f : S^1 \rightarrow S^1$ can be viewed as a continuous function $f : [0, 2\pi] \rightarrow S^1$ such that $f(0) = f(2\pi)$. Similarly, every continuous function $g : [0, 2\pi] \rightarrow S^1$ with the property $g(0) = g(2\pi)$ can be directly associated with a circle function $g : S^1 \rightarrow S^1$. Whether the domain is defined as $S^1$ or as $[0, 2\pi]$, the term circle function will be used in either case.

Definition 131 Circle functions $f, g : [0, 2\pi] \rightarrow S^1$ are circle homotopic if there is a homotopy $H : [0, 2\pi] \times [0, 1] \rightarrow S^1$ between $f$ and $g$ such that $H(0, t) = H(2\pi, t)$ for every $t \in [0, 1]$. $H$ is then called a circle homotopy.
**Remark 132** Every homotopy $G : S^1 \times [0,1] \to S^1$ naturally corresponds to a circle homotopy $G : [0,2\pi] \times [0,1] \to S^1$, and every circle homotopy $H : [0,2\pi] \times [0,1] \to S^1$ naturally corresponds to a homotopy $H : S^1 \times [0,1] \to S^1$ whose domain is $S^1$.

For the remainder of the appendix, we shall refer to the function $p : \mathbb{R} \to S^1$, defined by $p(\theta) = e^{i\theta}$, as the “projection”. Notice that $p([r,r+2\pi])$ is a bijection which projects $[r,r+2\pi)$ onto $S^1$ for any $r \in \mathbb{R}$, and that $p((a,b))$ is a homeomorphism onto its image in $S^1$, so long as $b-a \leq 2\pi$. Thus, if we restrict $p$ to the open interval $(r,r+2\pi)$, we will refer to the inverse of $p|_{(r,r+2\pi)}$ as $q_r : S^1 - \{e^{ir}\} \to \mathbb{R}$ (see Figure 28). Hence, $q_r$ is a homeomorphism which maps the circle with the point $e^{ir}$ removed to the open interval $(r,r+2\pi)$, essentially “unwrapping” the circle. With this in mind, we may now proceed to the next definition.

![Figure 28: The functions $q_r$ and $p|_{(r,r+2\pi)}$ are inverse homeomorphisms.](image)

**Definition 133** Let $X$ be a topological space, $f : X \to S^1$ be continuous, and let $p : \mathbb{R} \to S^1$ be the projection. A function $f^* : X \to \mathbb{R}$ is called a lifting of $f$ if $f^*$ is continuous and $p \circ f^* = f$.

The next theorem follows immediately from the previous definition. Here $\subset$ denotes a proper subset; i.e., $A \subset B$ implies $A \neq B$.

**Theorem 134** Let $f : X \to S^1$ be continuous, and suppose that $f$ is surjective, i.e., $f(X) \subset S^1$. Also, assume that $x_0 \in X$ and $r_0 \in p^{-1}(f(x_0))$. Then there exists a lifting $f^* : X \to \mathbb{R}$ of $f$ such that $f^*(x_0) = r_0$. 

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Proof. Since \( f(X) \subset S^1 \) and \( f(X) \neq S^1 \), there exists an \( s \in S^1 \) such that \( s \notin f(X) \). Now, choose an \( r \in p^{-1}(s) \) such that \( r_0 \in (r, r+2\pi) \). Define \( f^* : X \to \mathbb{R} \) by \( f^*(x) = q_r(f(x)) \), where 
\[
q_r : S^1 - \{e^{ir}\} \to \mathbb{R}
\]
R is the homeomorphism mentioned earlier. Then \( f(x) = p(q_r(f(x))) = p(f^*(x)) \), making \( f^* \) a lifting of \( f \) by Definition 133. Furthermore, \( r_0 \in p^{-1}(f(x_0)) \), \( q_r = p^{-1}|_{(r, r+2\pi)} \), and \( r_0 \in (r, r + 2\pi) \) together imply that \( f^*(x_0) = q_r(f(x_0)) = r_0 \).

**Theorem 135** Let \( X \) be connected, and let \( f : X \to S^1 \) be continuous. If \( g, h : X \to \mathbb{R} \) are liftings of \( f \), then there exists an \( n \in \mathbb{Z} \) such that \( g(x) - h(x) = 2\pi n \) for every \( x \in X \).

**Proof.** Consider an arbitrary \( x_0 \in X \). Because \( g \) and \( h \) are liftings of \( f \), \( p(g(x_0)) = p(h(x_0)) = f(x_0) \). Now, for all \( n \in \mathbb{Z} \),
\[
\begin{align*}
  e^{i(\theta + 2n\pi)} &= e^{i\theta} e^{2n\pi i} \\
  &= e^{i\theta} (e^{i\pi})^{2n} \\
  &= e^{i\theta} (-1)^{2n} \\
  &= e^{i\theta}.
\end{align*}
\]
This implies that \( p(\theta) = p(\theta + 2n\pi) \) for any integer \( n \). Thus, for some \( n_x \in \mathbb{Z}, \ g(x_0) = h(x_0) + 2\pi n_x \implies g(x_0) - h(x_0) = 2\pi n_x \). Define \( k : X \to \mathbb{R} \) by \( k(x) = g(x) - h(x) \). Clearly, \( k(X) = \{2\pi n_x | x \in X\} \). But, since \( g \) and \( h \) are continuous, \( k \) must also be continuous, implying that \( k(X) \) is connected in \( \mathbb{R} \). Therefore, \( n_x = n_y \) for all \( x, y \in X \), else \( \{2\pi n_x | x \in X\} \) would be disconnected. Hence, for some \( n \in \mathbb{Z} \), \( g(x) - h(x) = 2\pi n \) for all \( x \in X \).

**Corollary 136** Let \( f : X \to S^1 \) be continuous, \( X \) be connected, and \( g, h : X \to \mathbb{R} \) be liftings of \( f \). Suppose there exists some \( x_0 \in X \) such that \( g(x_0) = h(x_0) \). Then \( g(x) = h(x) \) for all \( x \in X \).

**Proof.** Due to our hypotheses, \( g(x) - h(x) = 2\pi n \) for some \( n \in \mathbb{Z} \) by Theorem 135. Thus, if \( g(x_0) = h(x_0) \) for some \( x_0 \in X \), then \( n = 0 \). Consequently, \( g(x) - h(x) = 0 \implies g(x) = h(x) \) for all \( x \in X \).
A.5 Every Circle Function Has a Lifting; Circle Functions Homotopic to $c_n$

**Theorem 137** Let $f : [0, 2\pi] \to S^1$ be continuous. Then there exists a function $f^* : [0, 2\pi] \to \mathbb{R}$ that is a lifting of $f$.

**Proof.** The complex number $e^{i0} = 1$ and $e^{i\pi} = -1$. Now, let $U = S^1 - \{1\}$ and $V = S^1 - \{-1\}$. $U$ and $V$ are both open, and together they cover $S^1$. Thus, since $f$ is continuous, $\mathcal{O} = \{f^{-1}(U), f^{-1}(V)\}$ is an open covering of $[0, 2\pi]$. By **Corollary 129**, there exists a subdivision $0 = \theta_0 < \theta_1 < \ldots < \theta_n = 2\pi$ of $[0, 2\pi]$ such that for all $j = 1, 2, \ldots, n$, the interval $[\theta_{j-1}, \theta_j]$ is a subset of either $f^{-1}(U)$ or $f^{-1}(V)$. For each interval $[\theta_{j-1}, \theta_j]$, the restriction $f|_{[\theta_{j-1}, \theta_j]}$ is not surjective, so **Theorem 134** tells us that a lifting of $f|_{[\theta_{j-1}, \theta_j]}$ exists. We will use these “restricted liftings” to construct a lifting $f^* : [0, 2\pi] \to \mathbb{R}$.

First, let $g_1 : [\theta_0, \theta_1] \to \mathbb{R}$ be a lifting of $f|_{[\theta_0, \theta_1]}$. Set $f_1^*$, the first part of our lifting, equal to $g_1$. By definition, $p(g_1(\theta_1)) = f(\theta_1)$, so $g_1(\theta_1) = p^{-1}(f(\theta_1))$. Similarly, by **Theorem 134**, there exists a lifting $g_2 : [\theta_1, \theta_2] \to \mathbb{R}$ of $f|_{[\theta_1, \theta_2]}$ such that $g_2(\theta_1) = p^{-1}(f(\theta_1)) = g_1(\theta_1)$. It follows from the Pasting Lemma (**Lemma 79**) that the function $f_2^* : [\theta_0, \theta_2] \to \mathbb{R}$, constructed by “pasting” $g_1$ and $g_2$ together, is continuous and, in fact, a lifting of $f|_{[\theta_0, \theta_2]}$. We can continue to construct liftings $f_i^*$ in this manner until the $n$th step, at which point $f_n^* : [\theta_0, \theta_n] \to \mathbb{R}$ is the desired lifting $f^* : [0, 2\pi] \to \mathbb{R}$. □

As was mentioned at the beginning of the appendix, our next theorem will prove that every circle function $f$ is homotopic to a “$c_n$” function.

**Theorem 138** Let $f : [0, 2\pi] \to S^1$ be a circle function. Then there exists an $n \in \mathbb{Z}$ such that $f$ is circle homotopic to $c_n$.

**Proof.** By **Theorem 137**, there exists a lifting $f^* : [0, 2\pi] \to \mathbb{R}$ of $f$. Because $f$ is a circle function, $f(0) = f(2\pi)$, which implies that $p(f^*(0)) = p(f^*(2\pi))$. Therefore, there is an $n \in \mathbb{Z}$ such that $f^*(2\pi) = f^*(0) + 2\pi n \implies f^*(2\pi) - f^*(0) = 2\pi n$.

Let us consider the homotopy $H : [0, 2\pi] \times [0, 1] \to \mathbb{R}$ defined by

$$H(\theta, t) = (1 - t)f^*(\theta) + tn\theta.$$ 

Notice that $H$ is a linear homotopy from $f^*$ to the function $c_n^* : [0, 2\pi] \to \mathbb{R}$ defined by $c_n^*(\theta) = n\theta$. We want a circle homotopy from $f$ to the circle function $c_n : [0, 2\pi] \to S^1$
defined by $c_n(\theta) = e^{in\theta}$.

Define $G : [0, 2\pi] \times [0, 1] \to S^1$ by $G(\theta, t) = p(H(\theta, t))$, and consider an arbitrary $t_0 \in [0, 1]$. Now, notice that $H(0, t_0) = (1 - t_0)f^*(0)$, and

\[
H(2\pi, t_0) = (1 - t_0)f^*(2\pi) + tn(2\pi)
= (1 - t_0)(f^*(0) + 2\pi n) + t_02\pi n
= 2\pi n - t_02\pi n + (1 - t_0)f^*(0) + t_02\pi n
= (1 - t_0)f^*(0) + 2\pi n
= H(0, t_0) + 2\pi n.
\]

It is clear that $p(H(2\pi, t_0)) = p(H(0, t_0) + 2\pi n) = p(H(0, t_0))$. Therefore, $G(0, t) = G(2\pi, t)$ for all $t \in [0, 1]$ since $t_0$ was arbitrary. Combined with the fact that $G$ is continuous, this verifies that $G$ is a circle homotopy. Finally, to show that $G$ is the desired circle homotopy,

\[
G(\theta, 0) = p(f^*(\theta)) = f(\theta),
\]

and,

\[
G(\theta, 1) = p(n\theta) = e^{in\theta} = c_n(\theta).
\]

\[\square\]

A.6 Every Circle Homotopy Has a Lifting; Circle Function Homotopy Classes Defined by $c_n$

Theorem 139 Let $H : [0, 2\pi] \times [0, 1] \to S^1$ be continuous. Then there exists a function $H^* : [0, 2\pi] \times [0, 1] \to \mathbb{R}$ that is a lifting of $H$.

The proof of Theorem 139 is very similar to that of Theorem 137. The difference is that, instead of pasting together liftings of subintervals of $[0, 2\pi]$, we paste together liftings of subrectangles of $[0, 2\pi] \times [0, 1]$. Since we have already seen the process carried out once, the formal proof of Theorem 139 will be omitted from the paper. Instead, the process is depicted informally in Figure 29.

Now we are finally able to verify the uniqueness of the $n$ whose existence we proved in Theorem 138.
Theorem 140  If $c_n$ is circle homotopic to $c_m$, then $n = m$.

Proof. If $c_n$ is circle homotopic to $c_m$, then there exists a circle homotopy $G : [0, 2\pi] \times [0, 1] \to S^1$ with $G(\theta, 0) = c_n(\theta)$ and $G(\theta, 1) = c_m(\theta)$. Theorem 139 then tells us that there exists a lifting $G^* : [0, 2\pi] \times [0, 1] \to \mathbb{R}$ of $G$. It follows that $G^*(\theta, 0)$ is a lifting of $c_n(\theta)$. But, the function $c_n^* : [0, 2\pi] \to \mathbb{R}$ defined by $c_n^*(\theta) = n\theta$ is also a lifting of $c_n$. Therefore, Theorem 135 tells us that $G^*(\theta, 0) = n\theta + 2\pi a$ for some $a \in \mathbb{Z}$. Consequently,

$$G^*(2\pi, 0) - G^*(0, 0) = (n(2\pi) + 2\pi a) - (n(0) + 2\pi a)$$

$$= 2\pi n + 2\pi a - 2\pi a$$

$$= 2\pi n.$$

Because $G^*(\theta, 1)$ also happens to be a lifting of $c_m$, we obtain

$$G^*(2\pi, 1) - G^*(0, 1) = 2\pi m$$

in similar fashion. Yet, due to the fact that $G$ is a circle homotopy, we know that $G(0, t) = G(2\pi, t)$ for every $t \in [0, 1]$, which implies that $G^*(0, t)$ and $G^*(2\pi, t)$ are liftings of the same function. Theorem 135 then tells us that $G^*(2\pi, t) - G^*(0, t) = 2\pi b$ for some $b \in \mathbb{Z}$ and all $t \in [0, 1]$. Hence,

$$G^*(2\pi, 1) - G^*(0, 1) = 2\pi b = G^*(2\pi, 0) - G^*(0, 0) \implies 2\pi n = 2\pi m,$$

which proves that $n = m$ as required. ■
B  Fundamental Theorems of Plane Geometry

The following is a list of certain theorems from plane geometry that the reader may find useful. Many of the results in Section 5 are obtained by using theorems and/or definitions believed to be “common knowledge,” i.e., the definition of a circle, two points define a line, and others. Here I have submitted the results that are neither intuitively obvious nor common knowledge. The reader may find proofs of these theorems in an undergraduate geometry textbook such as [6]. They are not proven here because plane geometric proofs are not the focus of Section 5; rather, the focus is on the fixed points of isometries.

Theorem 141  Three non-collinear points define a triangle.

Theorem 142  (Triangle Inequality) Let \( \triangle ABC \) be a triangle. Then, without loss of generality, \( AB + AC > BC \).

Theorem 143  Side-Side-Side (SSS): If \( \triangle ABC \) and \( \triangle DEF \) are such that \( AB \cong DE \), \( BC \cong EF \), and \( AC \cong DF \), then \( \triangle ABC \cong \triangle DEF \).

Theorem 144  Corresponding Parts of Congruent Triangles are Congruent (CPCTC): Let \( \triangle ABC \cong \triangle DEF \), and let \( A \) correspond to \( D \), \( B \) correspond to \( E \), and \( C \) correspond to \( F \). Then \( AB \cong DE \), \( BC \cong EF \), \( AC \cong DF \), \( \angle A \cong \angle D \), \( \angle B \cong \angle E \), and \( \angle C \cong \angle F \).

Theorem 145  Let \( A,B \) be distinct points. There exists a unique midpoint \( M \) of \( AB \).

Theorem 146  Let \( A,B \) be distinct points. There exists a unique perpendicular bisector of \( AB \).

Theorem 147  Let \( A,B \) be distinct points, and let \( \ell \) be the perpendicular bisector of \( AB \). The line \( \ell \) is the set of all points which are equidistant from points \( A \) and \( B \).

Theorem 148  Let \( \ell,m \) be distinct lines and let points \( P,Q \in m \) be on the same side of \( \ell \), \( P \neq Q \). Then \( P \) and \( Q \) are equidistant from \( \ell \) if and only if \( \ell \parallel m \).

Theorem 149  Side-Angle-Side (SAS): If \( \triangle ABC \) and \( \triangle DEF \) are such that \( AB \cong DE \), \( \angle ABC \cong \angle DEF \), and \( BC \cong EF \), then \( \triangle ABC \cong \triangle DEF \).
Theorem 150 Consider $\triangle ABC$. The vertex $B$ is on the perpendicular bisector of $\overline{AC}$ if and only if $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{BC}$.

Theorem 151 Consider $\triangle ABC$. The perpendicular bisector of $\overline{AC}$ bisects $\angle ABC$ if and only if $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{BC}$.
References


