MA 514 Assignment 3. Due Friday October 13 at 12:50pm.

Problem X2.4. The breadth-first search algorithm provides a method for finding a shortest path between two vertices of G. If T is a tree rooted at the vertex v_0, then the *level* of a vertex $v \in V(T)$ is the length of the unique path between v_0 and v in T. In the following, let G be a finite connected graph. Let T be a breadth-first search tree for G, rooted at v_0.

(a) Prove that the vertices of G enter T in non-decreasing order of level.
(b) Prove that each non-tree edge of G joins vertices that are at most one level apart in T.
(c) Let $v \in V(G)$. Prove that the path from v_0 to v in T is a shortest path from v_0 to v in G.

Problem 3C. Show that equality cannot hold in

$$N(p, q; 2) \leq N(p-1, q; 2) + N(p, q-1; 2)$$

if both terms on the right-hand side are even.

Problem 3D. Prove that $N(4, 4; 2) = 18$.

Problem 3F. Apply Ramsey's Theorem to prove that for all $r \in \mathbb{N}$ there is a minimal number $N(r)$ with the following property. If $n \geq N(r)$ and the integers in $\{1, 2, \ldots, n\}$ are coloured with r colours, then there are three integers x, y, z (not necessarily distinct) with the same colour and $x + y = z$. Determine the exact value of $N(2)$.

Problem 3I. Construct an edge-colouring of K_{16} with the three colours $\{0, 1, 2\}$ so that there are no monochromatic triangles, by doing the following.

The finite field of order sixteen can be explicitly constructed as $\mathbb{F}_{16} = \mathbb{F}_2[x]/(x^4 + x + 1) \cong \mathbb{F}_2[\alpha]$, where $\alpha^4 = \alpha + 1$. Label the vertices of K_{16} by the elements of $\mathbb{F}_{16} = \{0, \alpha, \alpha^2, \ldots, \alpha^{14}, \alpha^{15} = 1\}$. If β and γ are two vertices and $\beta + \gamma = \alpha^\nu$, assign the colour $\nu \mod 3$ to the edge $\{\beta, \gamma\}$.

(a) Show that the sum of two cubes in $\mathbb{F}_{16}$ cannot be a cube.
(b) Deduce that there cannot be a monochromatic triangle of colour 0.
(c) Deduce that there cannot be a monochromatic triangle of colour 1 or 2.
(d) Conclude that $N(3, 3, 3; 2) = 17$.