Infinite Series

Series : Let (a_n) be a sequence of real numbers. Then an expression of the form $a_1 + a_2 + a_3 + ...$ denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

Examples : 1. $1 + \frac{1}{2} + \frac{1}{3} + ...$ or $\sum_{n=1}^{\infty} \frac{1}{n}$ 2. $1 + \frac{1}{4} + \frac{1}{9} + ...$ or $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Partial sums : $S_n = a_1 + a_2 + a_3 + ... + a_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$,

Convergence or Divergence of $\sum_{n=1}^{\infty} a_n$

If $S_n \to S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S. If (S_n) does not converge then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges because $S_n = log(n+1)$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges because $S_n = 1 - \frac{1}{n+1} \to 1$.
3. If $0 < x < 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ because $S_n = \frac{1-x^{n+1}}{1-x}$.

Necessary condition for convergence

Theorem 1 : *If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.*

Proof : $S_{n+1} - S_n = a_{n+1} \to S - S = 0$. □

The condition given in the above result is necessary but not sufficient i.e., it is possible that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. If $\mid x \mid \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges because $a_n \nrightarrow 0$.
2. $\sum_{n=1}^{\infty} sinn$ diverges because $a_n \nrightarrow 0$.
3. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges, however, $log(\frac{n+1}{n}) \to 0$.

Necessary and sufficient condition for convergence

Theorem 2: *Suppose $a_n \geq 0 \ \forall \ n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded above.*

Proof : Note that under the hypothesis, (S_n) is an increasing sequence. □

Example : The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because

$$S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + ... + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2}$$

for all k.

Theorem 3: *If $\sum_{n=1}^{\infty} \mid a_n \mid$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof : Since $\sum_{n=1}^{\infty} \mid a_n \mid$ converges the sequence of partial sums of $\sum_{n=1}^{\infty} \mid a_n \mid$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion. □

Remark : Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \geq 1$.

Tests for Convergence

Let us determine the convergence or the divergence of a series by comparing it to one whose behavior is already known.

Theorem 4 : (**Comparison test**) *Suppose* $0 \le a_n \le b_n$ *for* $n \ge k$ *for some* k. *Then*

(1) The convergence of $\sum_{n=1}^{\infty} b_n$ *implies the convergence of* $\sum_{n=1}^{\infty} a_n$.

(2) The divergence of $\sum_{n=1}^{\infty} a_n$ *implies the divergence of* $\sum_{n=1}^{\infty} b_n$.

Proof : (1) Note that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded. Apply Theorem 2.

(2) This statement is the contrapositive of (1). □

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges because $\frac{1}{(n+1)(n+1)} \leqslant \frac{1}{n(n+1)}$. This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leqslant \frac{1}{\sqrt{n}}$.
3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $n^2 < n!$ for $n \geqslant 4$.

Problem 1 : *Let* $a_n \ge 0$. *Then show that both the series* $\sum\limits_{n\ge1} a_n$ *and* $\sum\limits_{n\ge1} \frac{a_n}{a_n+1}$ *converge or diverge together.*

Solution : Suppose $\sum\limits_{n\ge1} a_n$ converges. Since $0 \le \frac{a_n}{1+a_n} \le a_n$ by comparison test $\sum\limits_{n\ge1} \frac{a_n}{1+a_n}$ converges.

Suppose $\sum\limits_{n\ge1} \frac{a_n}{1+a_n}$ converges. By the Theorem 1, $\frac{a_n}{1+a_n} \to 0$. Hence $a_n \to 0$ and therefore $1 \le 1 + a_n < 2$ eventually. Hence $0 \le \frac{1}{2} a_n \le \frac{a_n}{1+a_n}$. Apply the comparison test.

Theorem 5 : (**Limit Comparison Test**) *Suppose* $a_n, b_n \ge 0$ *eventually. Suppose* $\frac{a_n}{b_n} \to L$.

1. *If* $L \in \mathbb{R}, L > 0$, *then both* $\sum_{n=1}^{\infty} b_n$ *and* $\sum_{n=1}^{\infty} a_n$ *converge or diverge together.*
2. *If* $L \in \mathbb{R}, L = 0$, *and* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
3. *If* $L = \infty$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof: 1. Since $L > 0$, choose $\epsilon > 0$, such that $L - \epsilon > 0$. There exists n_0 such that $0 \le L - \epsilon < \frac{a_n}{b_n} < L - \epsilon$. Use the comparison test.

2. For each $\epsilon > 0$, there exists n_0 such that $0 < \frac{a_n}{b_n} < \epsilon$, $\forall\ n > n_0$. Use the comparison test.

3. Given $\alpha > 0$, there exists n_0 such that $\frac{a_n}{b_n} > \alpha\ \forall\ n > n_0$. Use the comparison test. □

Examples :

1. $\sum_{n=1}^{\infty} (1 - n sin\frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.
2. $\sum_{n=1}^{\infty} \frac{1}{n} log(1 + \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.

Theorem 6 (**Cauchy Test or Cauchy condensation test**) *If* $a_n \ge 0$ *and* $a_{n+1} \le a_n\ \forall\ n$, *then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{k=0}^{\infty} 2^k a_{2^k}$ *converges.*

Proof : Let $S_n = a_1 + a_2 + ... + a_n$ and $T_k = a_1 + 2a_2 + ... + 2^k a_{2^k}$.

Suppose (T_k) converges. For a fixed n, choose k such that $2^k \geq n$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + ... + a_n \\ &\leq a_1 + (a_2 + a_3) + ... + (a_{2^k} + ... + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + ... + 2^k a_{2^k} \\ &= T_k. \end{aligned}$$

This shows that (S_n) is bounded above; hence (S_n) converges.

Suppose (S_n) converges. For a fixed k, choose n such that $n \geq 2^k$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + ... + a_n \\ &\geq a_1 + a_2 + (a_3 + a_4) ..... + (a_{2^{k-1}+1} + ... + a_{2^k}) \\ &\geq \tfrac{1}{2} a_1 + a_2 + 2a_4 + ... + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2} T_k. \end{aligned}$$

This shows that (T_k) is bounded above; hence (T_k) converges. □

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(logn)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Problem 2 : Let $a_n \geq 0, a_{n+1} \leq a_n \; \forall \; n$ and suppose $\sum a_n$ converges. Show that $na_n \to 0$ as $n \to \infty$.

Solution : By Cauchy condensation test $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. Therefore $2^k a_{2^k} \to 0$ and hence $2^{k+1} a_{2^k} \to 0$ as $k \to \infty$. Let $2^k \leq n \leq 2^{k+1}$. Then $na_n \leq na_{2^k} \leq 2^{k+1} a_{2^k} \to 0$. This implies that $na_n \to 0$ as $n \to \infty$.

Theorem 7 (**Ratio test**) *Consider the series* $\sum_{n=1}^{\infty} a_n, \; a_n \neq 0 \; \forall \; n$.

1. *If* $\mid \frac{a_{n+1}}{a_n} \mid \leq q$ *eventually for some* $0 < q < 1$, *then* $\sum_{n=1}^{\infty} \mid a_n \mid$ *converges.*
2. *If* $\mid \frac{a_{n+1}}{a_n} \mid \geq 1$ *eventually then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof: 1. Note that for some N, $\mid a_{n+1} \mid \leq q \mid a_n \mid \;\; \forall \; n \;\; \geq N$. Therefore, $\mid a_{N+p} \mid \leq q^p \mid a_N \mid$ $\forall \; p > 0$. Apply the comparison test.

2. In this case $\mid a_n \mid \nrightarrow 0$.

Corollary 1: *Suppose* $a_n \neq 0 \; \forall \; n$, *and* $\mid \frac{a_{n+1}}{a_n} \mid \to L$ *for some* L.

1. *If* $L < 1$ *then* $\sum_{n=1}^{\infty} \mid a_n \mid$ *converges.*
2. *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $L = 1$ *we cannot make any conclusion.*

Proof :

1. Note that $\mid \frac{a_{n+1}}{a_n} \mid < L + \frac{(1-L)}{2}$ eventually. Apply the previous theorem.

2. Note that $\mid \frac{a_{n+1}}{a_n} \mid > L - \frac{(L-1)}{2}$ eventually. Apply the previous theorem.

Examples :

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \rightarrow 0$.

2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = (1+\frac{1}{n})^n \rightarrow e > 1$.

3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $\frac{a_{n+1}}{a_n} \rightarrow 1$.

Theorem 8 : (**Root Test**) *If* $0 \leq a_n \leq x^n$ *or* $0 \leq {a_n}^{1/n} \leq x$ *eventually for some* $0 < x < 1$ *then* $\sum_{n=1}^{\infty} \mid a_n \mid$ *converges.*

Proof : Immediate from the comparison test. □

Corollary 2: *Suppose* $\mid a_n \mid^{1/n} \rightarrow L$ *for some* L. *Then*

1. *If* $L < 1$ *then* $\sum_{n=1}^{\infty} \mid a_n \mid$ *converges.*

2. *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

3. *If* $L = 1$ *we cannot make any conclusion.*

Examples :

1. $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$ converges because $a_n^{1/n} = \frac{1}{logn} \rightarrow 0$.

2. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges because $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \rightarrow \frac{1}{e} < 1$.

3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $a_n^{1/n} \rightarrow 1$.

Theorem 9 : (**Leibniz test**) *If* (a_n) *is decreasing and* $a_n \rightarrow 0$, *then* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Proof : Note that (S_{2n}) is increasing and bounded above by S_1. Similarly, (S_{2n+1}) is decreasing and bounded below by S_2. Therefore both converge. Since $S_{2n+1} - S_{2n} = a_{2n+1} \rightarrow 0$, both (S_{2n+1}) and (S_{2n}) converge to the same limit and therefore (S_n) converges. □

Examples : $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{logn}$ converge.

Problem 3: *Let* $\{a_n\}$ *be a decreasing sequence,* $a_n \geq 0$ *and* $\lim_{n \rightarrow \infty} a_n = 0$. *For each* $n \in \mathbb{N}$, *let* $b_n = \frac{a_1+a_2+...+a_n}{n}$. *Show that* $\sum_{n \geq 1} (-1)^n b_n$ *converges.*

Solution : Note that $b_{n+1} - b_n = \frac{1}{n+1}(a_1 + a_2 + ... + a_{n+1}) - \frac{1}{n}(a_1 + ... + a_n) = \frac{a_{n+1}}{n+1} - \frac{(a_1+...+a_n)}{n(n+1)}$. Since (a_n) is decreasing, $a_1 + ... + a_n \geq na_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1} - a_n}{n+1} \leq 0$. Hence (b_n) is decreasing.

We now need to show that $b_n \rightarrow 0$. For a given $\epsilon > 0$, since $a_n \rightarrow 0$, there exists n_0 such that $a_n < \frac{\epsilon}{2}$ for all $n \geq n_0$.

Therefore, $|\frac{a_1 + \cdots + a_n}{n}| = |\frac{a_1+\cdots+a_{n_0}}{n} + \frac{a_{n_0+1}+\cdots+a_n}{n}| \leq| \frac{a_1+\cdots+a_{n_0}}{n} | + \frac{n-n_0}{n} \frac{\epsilon}{2}$. Choose $N \geq n_0$ large enough so that $\frac{a_1+\cdots+a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1+\cdots+a_n}{n} < \epsilon$. Hence, $b_n \rightarrow 0$. Use the Leibniz test for convergence.