A CANONICAL LIFT OF FROBENIUS IN MORAVA E-THEORY

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Abstract. We prove that the pth Hecke operator on the Morava E-cohomology of a space is congruent to the Frobenius mod p. This is a generalization of the fact that the pth Adams operation on the complex K-theory of a space is congruent to the Frobenius mod p. The proof implies that the pth Hecke operator may be used to test Rezk's congruence criterion.

1. Introduction

The pth Adams operation on the complex K-theory of a space is congruent to the Frobenius mod p. This fact plays a role in Adams and Atiyah's proof [AA66] of the Hopf invariant one problem. It also implies the existence of a canonical operation θ on $K^0(X)$ satisfying

$$\psi^p(x) = x^p + p\theta(x),$$

when $K^0(X)$ is torsion-free. This extra structure was used by Bousfield [Bou96] to determine the λ-ring structure of the K-theory of an infinite loop space. There are several generalizations of the pth Adams operation in complex K-theory to Morava E-theory: the pth additive power operation, the pth Adams operation, and the pth Hecke operator. In this note, we show that the pth Hecke operator is a lift of Frobenius.

In [Rez09], Rezk studies the relationship between two algebraic structures related to power operations in Morava E-theory. One structure is a monad $\mathbb{T}$ on the category of E_0-modules that is closely related to the free E_∞-algebra functor. The other structure is a form of the Dyer-Lashof algebra for E, called Γ. Given a Γ-algebra R, each element $\sigma \in \Gamma$ gives rise to a linear endomorphism Q_σ of R. He proves that a Γ-algebra R admits the structure of an algebra over the monad $\mathbb{T}$ if and only if there exists an element $\sigma \in \Gamma$ (over a certain element $\bar{\sigma} \in \Gamma/p$) such that Q_σ is a lift of Frobenius in the following sense:

$$Q_\sigma(r) \equiv r^p \mod pR$$

for all $r \in R$.

We will show that Q_σ may be taken to be the pth Hecke operator T_p as defined by Ando in [And95, Section 3.6]. We prove this by producing a canonical element $\sigma_{can} \in \Gamma$ lifting the Frobenius class $\bar{\sigma} \in \Gamma/p$ [Rez09, Section 10.3] such that $Q_{\sigma_{can}} = T_p$. This provides us with extra algebraic structure on torsion-free algebras over the monad $\mathbb{T}$ in the form of a canonical operation θ satisfying

$$T_p(r) = r^p + p\theta(r).$$

Let $\mathbb{G}_{E_0}$ be the formal group associated to E, a Morava E-theory spectrum. The Frobenius ϕ on E_0/p induces the relative Frobenius isogeny

$$\mathbb{G}_{E_0/p} \longrightarrow \phi^*\mathbb{G}_{E_0/p}$$

over E_0/p. The kernel of this isogeny is a subgroup scheme of order p. By a theorem of Strickland, this corresponds to an E_0-algebra map

$$\bar{\sigma}\colon E^0(B\Sigma_p)/I \longrightarrow E_0/p,$$

where I is the image of the transfer from the trivial group to Σ_p. This map further corresponds to an element in the mod p Dyer-Lashof algebra Γ/p. Rezk considers the set of E_0-module maps $[\bar{\sigma}] \subset \hom(E^0(B\Sigma_p)/I, E_0)$ lifting $\bar{\sigma}$.

Proposition 1.1. There is a canonical choice of lift $\sigma_{can} \in [\bar{\sigma}]$.

The construction of σ_{can} is an application of the formula for the $K(n)$-local transfer (induction) along the surjection from Σ_p to the trivial group [Gan06, Section 7.3].

Let X be a space and let

$$P_p/I\colon E^0(X) \longrightarrow E^0(B\Sigma_p)/I \otimes_{E_0} E^0(X)$$

be the pth additive power operation. The endomorphism $Q_{\sigma_{can}}$ of $E^0(X)$ is the composite of P_p/I with $\sigma_{can} \otimes 1$.

Proposition 1.2. For any space X, the following operations on $E^0(X)$ are equal:

$$Q_{\sigma_{can}} = (\sigma_{can} \otimes 1)(P_p/I) = T_p.$$

This has the following immediate consequence:

Corollary 1.3. Let X be a space such that $E^0(X)$ is torsion-free. There exists a canonical operation

$$\theta\colon E^0(X) \longrightarrow E^0(X)$$

such that, for all $x \in E^0(X)$,

$$T_p(x) = x^p + p\theta(x).$$

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2. Tools

Let E be a height n Morava E-theory spectrum at the prime p. We will make use of several tools that let us access E-cohomology. We summarize them in this section.

For the remainder of this paper, let $E(X) = E^0(X)$ for any space X. We will also write E for the coefficients E^0 unless we state otherwise.

Character theory, Hopkins, Kuhn, and Ravenel introduce character theory for $E(BG)$ in [HKR00]. They construct the rationalized Drinfeld ring C_0 and introduce a ring of generalized class functions taking values in C_0:

$$Cl_n(G, C_0) = \{C_0\text{-valued functions on conjugacy classes of map from } \mathbb{Z}_p^n \text{ to } G\}.$$

They construct a map

$$E(BG) \longrightarrow Cl_n(G, C_0)$$

and show that it induces an isomorphism after the domain has been base-changed to C_0 [HKR00, Theorem C]. When $n = 1$, this is a p-adic version of the classical character map from representation theory.

Good groups, A finite group G is good if the character map

$$E(BG) \longrightarrow Cl_n(G, C_0)$$

is injective. Hopkins, Kuhn, and Ravenel show that Σ_{p^k} is good for all k [HKR00, Theorem 7.3].

Transfer maps, It follows from a result of Greenlees and Sadofsky [GS96] that there are transfer maps in E-cohomology along all maps of finite groups. In [Gan06, Section 7.3], Ganter studies the case of the transfer from G to the trivial group and shows that there is a simple formula for the transfer on the level of class functions. Let

$$\mathrm{Tr}_{C_0}\colon Cl_n(G, C_0) \longrightarrow C_0$$

be given by the formula $f \mapsto \frac{1}{|G|}\sum_{\alpha} f(\alpha)$, where the sum runs over all of the maps $\alpha\colon \mathbb{Z}_p^n \to G$. Ganter shows that there is a commutative diagram

$$\begin{tikzcd} E(BG) \arrow[r, "\mathrm{Tr}_E"] \arrow[d] & E \arrow[d] \\ Cl_n(G, C_0) \arrow[r, "\mathrm{Tr}_{C_0}"] & C_0, \end{tikzcd}$$

in which the vertical maps are the character map.

Subgroups of formal groups, Let $\mathbb{G}_E = \mathrm{Spf}(E(BS^1))$ be the formal group associated to the spectrum E. In [Str98], Strickland produces a canonical isomorphism

$$\mathrm{Spf}(E(B\Sigma_{p^k})/I) \cong \mathrm{Sub}_{p^k}(\mathbb{G}_E),$$

where I is the image of the transfer along $\Sigma_{p^{k-1}}^{\times p} \subset \Sigma_{p^k}$ and $\mathrm{Sub}_{p^k}(\mathbb{G}_E)$ is the scheme that classifies subgroup schemes of order p^k in $\mathbb{G}_E$. We will only need the case $k = 1$.

The Frobenius class, The relative Frobenius is a degree p isogeny of formal groups

$$\mathbb{G}_{E/p} \longrightarrow \phi^*\mathbb{G}_{E/p},$$

where $\phi\colon E/p \to E/p$ is the Frobenius. The kernel of the map is a subgroup scheme of order p. Using Strickland's result, there is a canonical map of E-algebras

$$\bar{\sigma}\colon E(B\Sigma_p)/I \longrightarrow E/p$$

picking out the kernel. In [Rez09, Section 10.3], Rezk describes this map in terms of a coordinate and considers the set of E-module maps $[\bar{\sigma}] \subset \hom(E(B\Sigma_p), E)$ that lift $\bar{\sigma}$.

Power operations, In [GH04], Goerss, Hopkins, and Miller prove that the spectrum E admits the structure of an E_∞-ring spectrum in an essentially unique way. This implies a theory of power operations. These are natural multiplicative non-additive maps

$$P_m\colon E(X) \longrightarrow E(B\Sigma_m) \otimes_E E(X)$$

for all $m > 0$. For $m = p^k$, they can be simplified to obtain interesting ring maps by further passing to the quotient

$$P_{p^k}/I\colon E(X) \longrightarrow E(B\Sigma_{p^k}) \otimes_E E(X) \longrightarrow E(B\Sigma_{p^k})/I \otimes_E E(X),$$

where I is the transfer ideal that appeared above.

Hecke operators, In [And95, Section 3.6], Ando produces operations

$$T_{p^k}\colon E(X) \longrightarrow E(X)$$

by combining the structure of power operations, Strickland's result, and ideas from character theory. Let $\mathbb{T} = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, let $H \subset \mathbb{T}$ be a finite subgroup, and let D_∞ be the Drinfeld ring at infinite level so that $\mathrm{Spf}(D_\infty) = \mathrm{Level}(\mathbb{T}, \mathbb{G}_E)$ and $\mathbb{Q} \otimes D_\infty = C_0$. Ando constructs an Adams operation depending on H as the composite

$$\psi^H \colon E(X) \xrightarrow{P_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{H \otimes 1} D_\infty \otimes_E E(X).$$

He then defines the p^kth Hecke operator

$$T_{p^k} = \sum_{\substack{H \subset \mathbb{T} \\ |H| = p^k}} \psi^H$$

and shows that this lands in $E(X)$.

3. A canonical representative of the Frobenius class

We construct a canonical representative of the set $[\bar{\sigma}]$. The construction is an elementary application of several of the tools presented in the previous section.

We specialize the transfers of the previous section to $G = \Sigma_p$. Let

$$\mathrm{Tr}_E \colon E(B\Sigma_p) \longrightarrow E$$

be the transfer from Σ_p to the trivial group and let

$$\mathrm{Tr}_{C_0} \colon Cl_n(\Sigma_p, C_0) \longrightarrow C_0$$

be the transfer in class functions from Σ_p to the trivial group. This is given by the formula

$$\mathrm{Tr}_{C_0}(f) = \frac{1}{p!} \sum_{\alpha} f(\alpha),$$

where $\alpha \in \hom(\mathbb{Z}_p^n, \Sigma_p)$. There are p^n elements in $\hom(\mathbb{Z}_p^n, \Sigma_p)$, so

$$\mathrm{Tr}_E(1) = \frac{p^n}{p!}.$$

Recall that $\mathbb{T} = (\mathbb{Q}_p/\mathbb{Z}_p)^n$ and let $\mathrm{Sub}_p(\mathbb{T})$ be the set of subgroups of order p in $\mathbb{T}$.

Lemma 3.1. [Mar, Section 4.3.6] The restriction map along $\mathbb{Z}/p \subseteq \Sigma_p$ induces an isomorphism

$$E(B\Sigma_p) \xrightarrow{\cong} E(B\mathbb{Z}/p)^{\mathrm{Aut}(\mathbb{Z}/p)}.$$

After a choice of coordinate x,

$$E(B\Sigma_p) \cong E[y]/(yf(y)),$$

where the degree of $f(y)$ is

$$|\mathrm{Sub}_p(\mathbb{T})| = \frac{p^n - 1}{p - 1} = \sum_{i=0}^{n-1} p^i,$$

$f(0) = p$, and y maps to x^{p-1} in $E(B\mathbb{Z}/p) \cong E[\![x]\!]/[p](x)$.

Lemma 3.2. [Qui71, Proposition 4.2] After choosing a coordinate, there is an isomorphism

$$E(B\Sigma_p)/I \cong E[y]/(f(y)),$$

and the ring is free of rank $|\mathrm{Sub}_p(\mathbb{T})|$ as an E-module.

After choosing a coordinate, the restriction map $E(B\Sigma_p) \to E$ sends y to 0 and the map

$$E(B\Sigma_p) \to E(B\Sigma_p)/I$$

is the quotient by the ideal generated by $f(y)$.

Lemma 3.3. The index of the E-module $E(B\Sigma_p)$ inside $E \times E(B\Sigma_p)/I$ is p.

Proof. This can be seen using the coordinate. There is a basis of $E(B\Sigma_p)$ given by the set $\{1, y, \dots, y^m\}$, where $m = |\operatorname{Sub}_p(\mathbb{T})|$, and a basis of $E \times E(B\Sigma_p)/I$ given by

$$\{(1,0), (0,1), (0,y), \dots, (0, y^{m-1})\}.$$

By Lemma 3.1, the image of the elements $\{1, y, \dots, y^{m-1}, p - f(y)\}$ in $E(B\Sigma_p)$ is the set

$$\{(1,1), (0,y), \dots, (0, y^{m-1}), (0,p)\}$$

in $E \times E(B\Sigma_p)/I$. The image of y^m is in the span of these elements and the submodule generated by these elements has index p. □

Lemma 3.4. [Rez09, Section 10.3] In terms of a coordinate, the Frobenius class

$$\bar{\sigma} \colon E(B\Sigma_p)/I \longrightarrow E/p$$

is the quotient by the ideal (y).

Now we modify Tr_{C_0} to construct a map

$$\sigma_{can} \colon E(B\Sigma_p)/I \longrightarrow E.$$

By Ganter's result [Gan06, Section 7.3] and the fact that Σ_p is good, the restriction of Tr_{C_0} to $E(B\Sigma_p)$ is equal to Tr_E. It makes sense to restrict Tr_{C_0} to

$$E \times E(B\Sigma_p)/I \subset Cl_n(\Sigma_p, C_0).$$

Lemma 3.3 implies that this lands in $\frac{1}{p}E$. Thus we see that the target of the map

$$p! \operatorname{Tr}_{C_0} |_{E \times E(B\Sigma_p)/I}$$

can be taken to be E. We may further restrict this map to the subring $E(B\Sigma_p)/I$ to get

$$p! \operatorname{Tr}_{C_0} |_{E(B\Sigma_p)/I} \colon E(B\Sigma_p)/I \longrightarrow E.$$

From the formula for Tr_{C_0}, for $e \in E \subset E(B\Sigma_p)/I$, we have

$$p! \operatorname{Tr}_{C_0} |_{E(B\Sigma_p)/I}(e) = (p^n - 1)e.$$

Note that $p-1$ is a p-adic unit, so we may set

$$\sigma_{can} = \frac{p!}{p-1} \operatorname{Tr}_{C_0} \Big|_{E(B\Sigma_p)/I}.$$

There are several reasonable ways to normalize σ_{can}, we have chosen to divide by $p-1$ because there are $p-1$ maps in each nontrivial conjugacy class $[\alpha \colon \mathbb{Z}_p^n \to \Sigma_p]$. This normalization gives $\sigma_{can}(e) = |\operatorname{Sub}_p(\mathbb{T})| e$ for any $e \in E \subset E(B\Sigma_p)/I$. Another reason for this choice is explained in the next section.

We now show that σ_{can} fits in the diagram

$$\begin{tikzcd} & E \arrow[d] \\ E(B\Sigma_p)/I \arrow[ur, "\sigma_{can}"] \arrow[r, "\bar{\sigma}"'] & E/p, \end{tikzcd}$$

where $\bar{\sigma}$ picks out the kernel of the relative Frobenius.

Proposition 3.5. The map

$$\sigma_{can}\colon E(B\Sigma_p)/I \longrightarrow E$$

is a representative of Rezk's Frobenius class.

Proof. We may be explicit. Choose a coordinate so that the quotient map

$$q\colon E(B\Sigma_p) \longrightarrow E(B\Sigma_p)/I$$

is given by

$$q\colon E[y]/(yf(y)) \longrightarrow E[y]/(f(y)).$$

We must show that

$$E(B\Sigma_p)/I \xrightarrow{\sigma_{can}} E \xrightarrow{\text{mod } p} E/p$$

is the quotient by the ideal $(y) \subset E(B\Sigma_p)/I$.

There is a basis of $E(B\Sigma_p)$ (as an E-module) given by $\{1, y, \ldots, y^m\}$, where $m = |\operatorname{Sub}_p(\mathbb{T})|$. We will be careful to refer to the image of y^i in $E(B\Sigma_p)/I$ as $q(y^i)$. For the basis elements of the form y^i, where $i \neq 0$, the restriction map $E(B\Sigma_p) \to E$ sends y^i to 0. Thus

$$\operatorname{Tr}_E(y^i) = \operatorname{Tr}_{C_0}|_{E(B\Sigma_p)/I}(q(y^i)) \in E.$$

Now the definition of σ_{can} implies that $\sigma_{can}(q(y^i))$ is divisible by p. So

$$\sigma_{can}(q(y^i)) \equiv 0 \mod p.$$

It is left to show that, for e in the image of $E \to E(B\Sigma_p)/I$,

$$\sigma_{can}(e) \equiv e \mod p.$$

Since $\sigma_{can}(e) = |\operatorname{Sub}_p(\mathbb{T})|e$, the result follows from the fact that $|\operatorname{Sub}_p(\mathbb{T})| \equiv 1 \bmod p$. $\square$

4. The Hecke operator congruence

We show that the pth additive power operation composed with σ_{can} is the pth Hecke operator. This implies that the Hecke operator satisfies a certain congruence.

The two maps in question are the composite

$$E(X) \xrightarrow{P_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{\sigma_{can}\otimes 1} E(X)$$

and the Hecke operator T_p described in Section 2.

Proposition 4.1. The pth additive power operation composed with the canonical representative of the Frobenius class is equal to the pth Hecke operator:

$$(\sigma_{can} \otimes 1)(P_p/I) = T_p.$$

Proof. This follows in a straight-forward way from the definitions. Unwrapping the definition of the character map, the map σ_{can} is the sum of a collection of maps

$$E(B\Sigma_p)/I \longrightarrow C_0,$$

one for each subgroup of order p in $\mathbb{T}$ (or non-trivial conjugacy class $[\alpha\colon \mathbb{Z}_p^n \to \Sigma_p]$).

Any map

$$\alpha\colon \mathbb{Z}_p^n \to \Sigma_p$$

factors through $(\mathbb{Z}/p)^n$. We will refer to the induced map $(\mathbb{Z}/p)^n \to \Sigma_p$ as α as well. Given $[\alpha]$, the character map to the factor of class functions corresponding to $[\alpha]$ is the composite

$$E(B\Sigma_p) \xrightarrow{\alpha^*} E(B(\mathbb{Z}/p)^n) \to D_\infty \to C_0.$$

If α is surjective, then this induces

$$E(B\Sigma_p)/I \xrightarrow{\alpha^*} E(B(\mathbb{Z}/p)^n)/I_{tr} \to D_\infty \to C_0,$$

where $I_{tr} \subset E(B(\mathbb{Z}/p)^n)$ is the ideal generated by transfers from proper subgroups. It is standard that, as $[\alpha]$ varies, these are the maps that classify the subgroups of order p in $\mathbb{T}$. For completeness, note that the subgroup can be read off of $[\alpha]$. It is image of the Pontryagin dual of the surjective map

$$\alpha\colon \mathbb{Z}_p^n \to \operatorname{im}\alpha.$$

□

Since $\sigma_{can} \in [\bar{\sigma}]$, the following diagram commutes

$$\begin{tikzcd}
E(X) \arrow[r, "P_p"] & E(B\Sigma_p) \otimes_E E(X) \arrow[r] \arrow[d, "\mathrm{Res}\otimes 1"'] & E(B\Sigma_p)/I \otimes_E E(X) \arrow[r, "\sigma_{can}\otimes 1"] \arrow[d, "\bar{\sigma}\otimes 1"'] & E(X) \arrow[dl] \\
& E(X) \arrow[r] & E(X)/p &
\end{tikzcd}$$

and this implies that

$$(\sigma_{can} \otimes 1)(P_p/I)(x) \equiv x^p \mod p.$$

Corollary 4.2. For $x \in E(X)$, there is a congruence

$$T_p(x) \equiv x^p \mod p.$$

Let X be a space with the property that $E(X)$ is torsion-free. The corollary above implies the existence of a canonical function

$$\theta\colon E(X) \longrightarrow E(X)$$

such that

$$T_p(x) = x^p + p\theta(x).$$

Example 4.3. When $n = 1$ and E is p-adic K-theory, $\mathbb{G}_E$ is a height 1 formal group,

$$E(B\Sigma_p)/I$$

is a rank one E-module, and σ_{can} is an E-algebra isomorphism. The composite

$$E(X) \xrightarrow{P_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{\sigma_{can}\otimes 1} E(X)$$

is the pth unstable Adams operation. In this situation, the function θ is understood by work of Bousfield [Bou96].

Example 4.4. At arbitrary height, we may consider the effect of T_p on $z \in \mathbb{Z}_p \subset E$. Since T_p is a sum of ring maps

$$T_p(z) = |\operatorname{Sub}_p(\mathbb{T})|z.$$

This is congruent to z^p mod p.

Example 4.5. At height 2 and the prime 2, Rezk constructed an E-theory associated to a certain elliptic curve [Rez]. He calculated P_2/I, when $X = *$. He found that, after choosing a particular coordinate x,

$$E(B\Sigma_2)/I \cong \mathbb{Z}_2[\![u_1]\!][x]/(x^3 - u_1x - 2)$$

and

$$P_2/I\colon \mathbb{Z}_2[\![u_1]\!] \longrightarrow \mathbb{Z}_2[\![u_1]\!][x]/(x^3 - u_1x - 2)$$

sends $u_1 \mapsto u_1^2 + 3x - u_1 x^2$. In [Dri74, Section 4B], Drinfeld explains how to compute the ring that corepresents $\mathbb{Z}/2 \times \mathbb{Z}/2$-level structures. Note that in the ring

$$\mathbb{Z}_2[\![u_1]\!][y,z]/(y^3 - u_1 y - 2),$$

y is a root of $z^3 - u_1 z - 2$ and

$$\frac{z^3 - u_1 z - 2}{z - y} = z^2 + yz + y^2 - u_1.$$

Drinfeld's construction gives

$$D_1 = \Gamma \operatorname{Level}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{G}_E) \cong \mathbb{Z}_2[\![u_1]\!][y,z]/(y^3 - u_1 y - 2, z^2 + yz + y^2 - u_1).$$

The point of this construction is that $x^3 - u_1 x - 2$ factors into linear terms over this ring. In fact,

$$x^3 - u_1 x - 2 = (x - y)(x - z)(x + y + z).$$

The three maps $E(B\Sigma_2)/I \to D_1 \subset C_0$ that show up in the character map are given by sending x to these roots. We see that

$$\sigma_{can}(x) = y + z - (y + z) = 0$$

and that

$$\begin{aligned} T_p(u_1) &= (\sigma_{can} \otimes 1)(P_2/I)(u_1) \\ &= 3u_1^2 - 2u_1 y^2 - 2u_1 z^2 - 2u_1 yz \\ &= u_1^2. \end{aligned}$$

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