The character of the total power operation. Jt w/ Barthel ①

Fix a prime $p$.

Let $E_n$ be Morava $E$-thy. $(E = E_n)$

$E_n^0 \cong W(k)[[u_1, \dots, u_{n-1}]]$ the Lubin-Tate ring.

$E_n \rightsquigarrow$ htn fg. $\mathbb{G}$.

Thm (Goerss-Hopkins-Miller) $E_n$ has an essentially unique $E_\infty$-ring structure.

In the homotopy category - Power operations.

$\mathbb{P}_m : E_n^0(X) \longrightarrow E_n^0(E\Sigma_m \times_{\Sigma_m} X^m)$ multiplicative, not additive

Idea: $X \to E_n \rightsquigarrow E\Sigma_m \times_{\Sigma_m} X^m \to E\Sigma_{m+} \wedge_{\Sigma_m} E_n^{\wedge m} \longrightarrow E_n$.

Quite mysterious: explicitly computed for $p=2$ $n=2$ $X=*$ by Rezk

$p=3$ $n=2$ $X=*$ by Zhu.

Goal: Give a formula for the "rationalization" of ~~this map~~ $\mathbb{P}_m$ using HKR character thy when $X = BG$. (finite gp)

Note: when $X = BG$ $E\Sigma_m \times_{\Sigma_m} BG^{\times m} = BG \wr \Sigma_m$..

Initial Goal:

$$\begin{tikzcd}
E_n(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"] & E(BG \wr \Sigma_m) \arrow[d, "\chi"] \\
Cl_n(G) \arrow[r, dashed, "?"] & Cl_n(G \wr \Sigma_m)
\end{tikzcd}$$

HKR Character Thy.

Let $\mathbb{L} = \mathbb{Z}_p^n$

$\hom(\mathbb{L}, G)$ n-tuples of commuting prime power order elements in $G$.

$G$ acts by conjugation

Let $\mathbb{T} = \mathbb{L}^* = \hom(\mathbb{L}, \mathbb{Q}_p/\mathbb{Z}_p)$, then $\mathbb{T} \cong \mathbb{Q}_p/\mathbb{Z}_p^n$.

HKR construct a ~~[illegible]~~ $p^{-1}E_n^0$-alg $C_0$ such that

$$\hom(C_0, R) = \mathrm{Iso}(\mathbb{T}, \mathbb{G}).$$

Define $Cl_n(G, C_0) = Cl_n(G) =$ $C_0$-valued fncs on $\hom(\mathbb{L}, G)/\sim$

Thm (HKR)

$$C_0 \otimes_E E(BG) \xrightarrow{\cong} Cl_n(G, C_0)$$

$Aut(\mathbb{T})$ acts on $C_0$ by precomposition $\quad C_0^{Aut(\mathbb{T})} = p^{-1}E^0$

$Aut(\mathbb{T})$ acts on $hom(\mathbb{L}, G)/\sim$ by precomp. w/ the dual.

Thm (HKR)

$$p^{-1}(E(BG)) \xrightarrow{\cong} Cl_n(G, C_0)^{Aut(\mathbb{T})}.$$

Constructing $P_m : Cl_n(G) \longrightarrow Cl_n(G \wr \Sigma_m)$

Prop: The action of $Aut(\mathbb{T})$ on $C_0$ extends to an action of $Isog(\mathbb{T})$.

Idea: Let $\phi_H : \mathbb{T} \longrightarrow \mathbb{T}$ be an isogeny w/ kernel $H \leq \mathbb{T}$.

$$\begin{tikzcd}
 & \mathbb{T} \arrow[r, "\cong"] \arrow[dl, "\phi_H"'] \arrow[d, "q_H"] & \mathbb{G} \arrow[d] \\
\mathbb{T} & \mathbb{T}/H \arrow[l, "\cong", "\psi_H"'] \arrow[r, "\cong"] & \mathbb{G}/H
\end{tikzcd}$$

sublattice

So $q_H^* : \mathbb{L}_H \hookrightarrow \mathbb{L}$, define $Cl_n^H(G, C_0) = C_0$-valued fncs on $hom(\mathbb{L}_H, \mathbb{G})/\sim$.

Can build a ring map $\phi_H : Cl_n(G, C_0) \longrightarrow Cl_n^H(G, C_0)$.

$$f \longmapsto f^{\phi_H}$$

where $f^{\phi_H}([\alpha]) = \phi_H^* f([\alpha \psi_H^*])$

Fact: This "extends" the action of $Aut(\mathbb{T})$ on $Cl_n(G)$.

Group Thy: ~~Some~~ bijections

$$hom(\mathbb{L}, \Sigma_m)/\sim \;\cong\; Sum_m(\mathbb{T})$$

$$hom(\mathbb{L}, \Sigma_m)/\sim \;\cong\; Sum_m(\mathbb{T}, G).$$

where $\mathrm{Sum}_m(\Pi) = \{ \bigoplus H_i \mid H_i \in \Pi, \sum |H_i| = m \}$

$$\mathrm{Sum}_m(\Pi, G) = \{ \bigoplus (H_i, [\alpha_i]) \mid [\alpha_i : \mathbb{L}_{H_i} \to G] \}$$

( Say "when $G = e \ldots$", "$|H_i| = p^k \ldots$" )

Idea:

$$\begin{tikzcd} & \prod \Sigma_{m_i} \arrow[d] \\ \mathbb{L} \arrow[r] \arrow[ur, dashed, "\prod \beta_i"] & \Sigma_m \end{tikzcd}$$

$\beta_i$ transitive

$H_i = \text{image} ( \mathrm{im} \beta_i^* \longrightarrow \mathbb{L}^* = \Pi )$.

Finally:

$$\begin{tikzcd} \mathrm{Isog}(\Pi) \arrow[d, "\text{kernel}"'] \\ \mathrm{Sub}(\Pi) \arrow[u, dashed, bend right, "\phi"'] \end{tikzcd}$$

$$\mathbb{P}_m^{\phi} : Cl_n(G) \longrightarrow Cl_n(G \wr \Sigma_m)$$
$$f \longmapsto \mathbb{P}_m^{\phi}(f)$$

$$\mathbb{P}_m^{\phi}(f) ( \bigoplus_i (H_i, [\alpha_i])) = \prod_i f^{\phi_{H_i}}([\alpha_i])$$
$$= \prod_i \phi_{H_i}^* f ([\alpha_i \psi_{H_i}^*]).$$

Thm: For every section $\phi$,

$$\begin{tikzcd} E(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E(BG \wr \Sigma_m) \arrow[d, "\chi"] \\ Cl_n(G) \arrow[r, "\mathbb{P}_m^{\phi}"] & Cl_n(G \wr \Sigma_m) \end{tikzcd}$$

commutes.

Sanity Check: $J \subseteq Cl_n(G \wr \Sigma_m)$

Demand more: $\mathbb{P}^{\phi}$ is a global power functor if

for $ij = m$

$$\begin{tikzcd} Cl_n(G) \arrow[r, "\mathbb{P}_m^{\phi}"] \arrow[d, "\mathbb{P}_i^{\phi}"'] & Cl_n(G \wr \Sigma_m) \arrow[d, "\nabla^*"] \\ Cl_n(G \wr \Sigma_i) \arrow[r, "\mathbb{P}_j^{\phi}"] & Cl_n(G \wr \Sigma_i \wr \Sigma_j) \end{tikzcd}$$

commutes

where $\nabla : \Sigma_i \wr \Sigma_j \to \Sigma_m$

( Say "like asking that $\mathbb{P}^{\phi}$ be a total power op" )

Def: A section $\phi$ is a power section if for $K \subseteq H \subseteq \mathbb{T}$

$$\phi_H = \phi_{H/K}\,\phi_K \quad \text{where } H/K = \phi_K(H).$$

Prop: $\mathbb{P}^\phi$ is a global power functor iff $\phi$ is a power section.

Ex.

$$\begin{tikzcd} \mathrm{Isog}(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[d, shift right] & {[p^k]} \\ \mathrm{Sub}(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[u, dashed, shift right] & \mathbb{Z}/p^k \arrow[u, mapsto] \end{tikzcd}$$

Ex. When $n=2$ we have constructed explicit examples.

Question: $n>2$?

Recall:

$$\begin{tikzcd} \mathbb{P}^\phi_m : Cl_n(G) \arrow[loop above, "\mathrm{Aut}(\mathbb{T})"] \arrow[r] & Cl_n(G \wr \Sigma_m) \arrow[loop above, "\mathrm{Aut}(\mathbb{T})"] \end{tikzcd}$$

Thm: $\mathbb{P}^\phi_m$ passes to $\mathrm{Aut}(\mathbb{T})$-invariants, and the resulting multiplicative natural transformation

$$p^{-1}\mathbb{P}_m : p^{-1}E(BG) \longrightarrow p^{-1}E(BG \wr \Sigma_m)$$

is independent of the choice of $\phi$.

("pf is a calculation…")

We have constructed a "rationalized total power operation"

$$\begin{tikzcd} E(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E(BG \wr \Sigma_m) \arrow[d, "\chi"] \\ p^{-1}E(BG) \arrow[r, "p^{-1}\mathbb{P}_m"] & p^{-1}E(BG \wr \Sigma_m) \end{tikzcd}$$

Thm: For all $n$, $p^{-1}\mathbb{P}$ is a global power functor.

("Two pfs.")

Applications: Compute everything.

① $P_m : E(BG) \longrightarrow E(BG\wr\Sigma_m) \longrightarrow E(BG)\otimes E(\Sigma_m)$

$P_m^\phi : Cl_n(G) \xrightarrow{\mathbb{P}_m^\phi} Cl_n(G\wr\Sigma_m) \longrightarrow Cl_n(G)\otimes Cl_n(\Sigma_m)$

$$G\wr\Sigma_m \xleftarrow{\Delta} G\times\Sigma_m$$

$$\hom(\mathbb{L}, G\wr\Sigma_m)/_{\sim} \longleftarrow \hom(\mathbb{L}, G\times\Sigma_m)/_{\sim}$$

$$\bigoplus_i (H_i, [\alpha q_{H_i}^*]) \hookleftarrow\!\shortmid \Big(\bigoplus_i H_i, [\alpha]\Big)$$

so $P_m^\phi(f)\big(\bigoplus_i H_i, [\alpha]\big) = \mathbb{P}_m^\phi\big(\bigoplus_i (H_i, [\alpha q_{H_i}^*])\big)$

$$= \prod_i \phi_{H_i}^* f([\alpha q_{H_i}^* \varphi_{H_i}^*])$$

$$= \prod_i \phi_{H_i}^* f([\alpha \phi_{H_i}^*]).$$

② $\psi_{p^k}$ ~~$P^\phi_{p^{kn}}$~~ $: Cl_n(G) \xrightarrow{P_{p^{kn}}^\phi} Cl_n(G)\otimes_{C_0} Cl_n(\Sigma_{p^{kn}}) \xrightarrow{\pi[p^k]} Cl_n(G)\otimes_{C_0} C_0 = Cl_n(G)$

If $\phi_{\pi[p^k]} = [p^k]$, $[p^k]^* : \mathbb{L} \xrightarrow{p^k} \mathbb{L}$

$$\psi_{p^k}(f)([\alpha]) = P_{p^{kn}}^\phi(f)(\pi[p^k], [\alpha])$$

$$= [p^k]^* f([\alpha [p^k]^*])$$

$$= f([\mathbb{L} \xrightarrow{p^k} \mathbb{L} \xrightarrow{\alpha} G]).$$

the expected formula.

$\psi_{p^k} : Cl_n(G) \to Cl_n(G)$

$$\psi_{p^k}(f)([g_1, \dots, g_n])$$

$$\rotatebox{-60}{=}$$

$$f([g_1^{p^k}, \dots, g_n^{p^k}])$$