Math 114 - Exam I Review

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February 5, 2018

Unit I: A Toolbox for Integral Calculus

Reminders

- There is a **review session** tonight, February 5, 6:00-8:00 PM, in FB 200 open to all students in MA 114.
- Exam I takes place next Tuesday, February 6 at 5:00 PM - see the course website for room assignments. Be sure to arrive ten minutes early and bring your student ID, a one-page "cheat sheet," an allowed calculator, pencils, and erasers.
- There will be no quiz in recitation this week. Your exams will be graded on Wednesday and should be returned to you in recitation on Thursday. There will be a new version of Worksheet 9 to download for Thursday's recitation. We will not be using the original worksheets 7 and 9.
- Remember that you have Webwork A6 on sequences due this Friday, and Webwork B1 on sequences by recursion due this coming Monday.

Today's REEF Polling Question

Are You Here Today?

A. Yes
B. No

Exam Overview

You will be tested on:

1. Techniques of Integration
   - Integration by parts
   - Special Trig Integrals
   - Trig Substitution (the triangle picture is your friend)
   - Integration by Partial Fractions
2. Numerical Integration
   - Left, Right, Midpoint, Trapezoid, Simpson
   - Error estimates
3. Improper Integrals (Type I and Type II, and Comparison Test)

Reminders - Two Kinds of Integrals

There are two types of integrals:

- *Indefinite* integrals, where the solution is a function of x, like

$$\int x \sin x \, dx = \sin(x) - x\cos(x) + C$$

- *Definite* integrals, where the the solution is a number, like

$$\int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$$

Reminders - Two Ways to Use u-Substitution

- For *indefinite* integrals, integration by substitution is a two-way trip: first convert to u, then compute the integral, then convert back to a function of x.

  Example: $\int 2xe^{-x^2}\, dx$

- For *definite* integrals, integration by substitution is a one-way trip: you change both the *integrand* and the *limits of integration* to the u variable and the corresponding values of u, and *never go back!*

  Example: $\int_0^{\pi/2} \sin^2(x)\cos(x)\, dx$

Which Method for Which Problem?

Which of the methods (integration by substitution, integration by parts, special trig methods, trig substitution, partial fractions) would you use to compute each of the following integrals? How would you start?

1. $\int x \ln x \, dx$
2. $\int \frac{7x^2 + x + 5}{x(x^2+1)} \, dx$
3. $\int \sin^2 x \cos^5 x \, dx$
4. $\int \frac{x^2 - 4}{x} \, dx$
5. $\int \frac{x}{x^2+1} \, dx$

Integration by Parts

$$\int x \ln x \, dx$$

Choose $u(x)$ and $v(x)$ to use the formula

$$\int u \, dv = uv - \int v \, du, \qquad dv = v'(x)\, dx, \quad du = u'(x)\, dx$$

You want v' to be easy to integrate and u to have a simple derivative

Another example: $\int x \arctan(x)\, dx$

Special Trig Integral Techniques

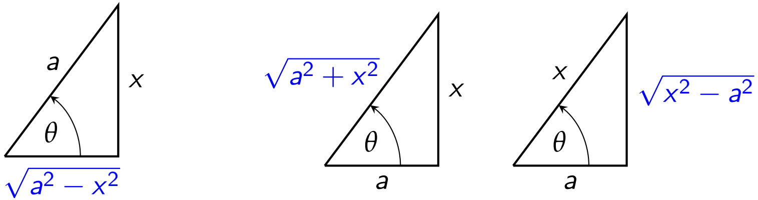
$$\int \sin^2 x \cos^5 x \, dx$$

1. Use known antidifferentiation formulas (what is $\int \sec^2 x \, dx$?)
2. Use u substitution when there is an extra power of sine or cosine (what is $\int \sin^2 x \cos^5 x \, dx$, in fact?)
3. Sometimes it helps to remember that $\tan x = \sin x / \cos x$, etc. – express everything in terms of sine and cosine

Trig Substitution

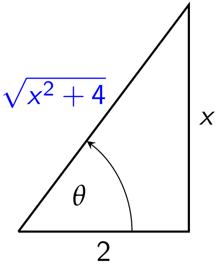
This method can be used for integrals that involve $x^2 + a^2$, $a^2 - x^2$, and $x^2 - a^2$ for some number a

Remember the “triangle pictures”



Trig Substitution

How do you find $\int \frac{x^2}{(x^2+4)^2}\,dx$?

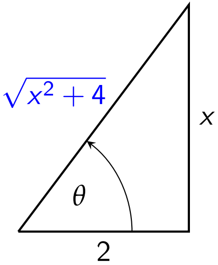


Trig Substitution

How do you find $\int \frac{x^2}{(x^2+4)^2}\,dx$?

$$\tan\theta = \frac{x}{2}$$

$$\sec\theta = \frac{\sqrt{x^2+4}}{2}$$



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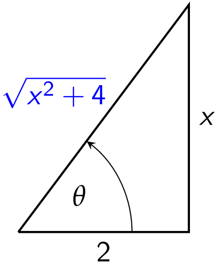
$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta,\, d\theta$$
$$\sqrt{x^2+4} = 2\sec\theta$$



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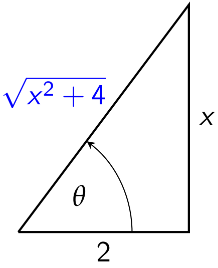

$$\int \frac{x^2}{(x^2+4)^2}\,dx = \int \frac{(2\tan\theta)^2}{(2\sec\theta)^4} 2\sec^2\theta\,d\theta$$

Integration by Partial Fractions

This method is used to integrate rational functions. To integrate $P(x)/Q(x)$ where P and Q are polynomials:

1. If the degree of P is larger than the degree of Q, divide P by Q to get a polynomial quotient plus a rational remainder
2. Factor the denominator into linear and irreducible quadratic factors
3. Compute the partial fraction decompostion
4. Integrate each 'building block'

Partial Fraction Decomposition

Know this table!

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Repeated Linear $(ax+b)^r$	$\frac{A_1}{ax-b}+\cdots+\frac{A_r}{(ax+b)^r}$
Irreducible Quadratic (ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
Repeated Irreducible Quadratic $(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Integration by Partial Fractions

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Irreducible Quadratic $(ax^2 + bx + c)$	$\frac{Ax+B}{ax^2+bx+c}$

Now let's integrate

$$\int \frac{7x^2 + x + 5}{x(x^2 + 1)}\, dx$$

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Now let's integrate

$$\int \frac{7x^2+x+5}{x(x^2+1)}\,dx$$

$$\frac{7x^2+x+5}{x(x^2+1)} = \frac{2x+1}{x^2+1} + \frac{5}{x}$$

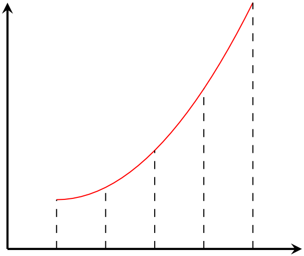
Numerical Integration

We studied increasingly accurate methods to compute the definite integral

$$\int_a^b f(x)\,dx$$

The first three are based on Riemann sums.

- The *left endpoint method*
- The *right endpoint method*
- The *midpoint method*



The last two use new ideas:

- The Trapezoid Method
- Simpson's Rule

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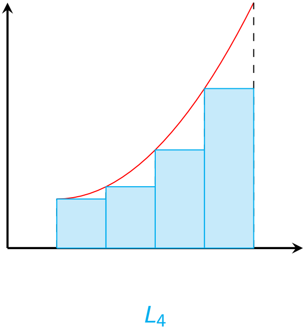

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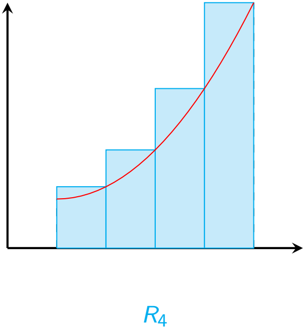

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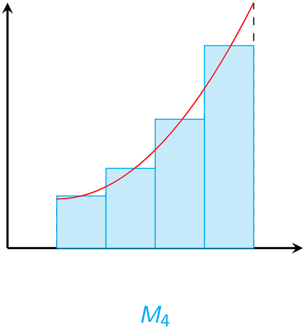

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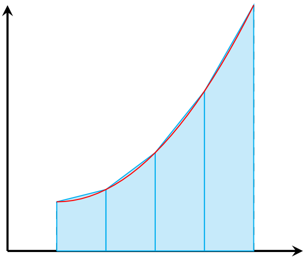

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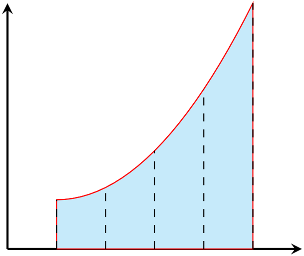

S_4

The last two use new ideas:

- The Trapezoid Method
- Simpson's Rule

Trapezoid and Simpson Rules

- *Trapezoid Rule* for $\int_a^b f(x)\,dx$ with n intervals

$$T_n = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

- *Simpson's Rule* for $\int_a^b f(x)\,dx$ with n intervals (n *even*)

$$S_n = \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \right.$$
$$\left. \ldots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

Trapezoid Rule and Simpson Rule Examples

- Trapezoid Rule for $\int_a^b f(x)\,dx$ with 5 intervals

$$T_5 = \frac{h}{2}\left[1f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 1f(x_5)\right]$$

- Simpson's Rule for $\int_a^b f(x)\,dx$ with 6 intervals

$$S_6 = \frac{h}{3}\left[1f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 1f(x_6)\right]$$

Using Error Estimates

The error $\left|\int_a^b f(x)\,dx - M_n\right|$ is denoted E_M, and the Midpoint Rule "warranty" guarantees that

$$E_T \leq \frac{K(b-a)^3}{24n^2}$$

where K is an upper bound on $|f''(x)|$ in $[a, b]$.

- How do you find K?
- How do use this warranty to choose n so that the error is less than say, 0.0001?

Improper Integrals

Improper integrals come in two kinds:

- Type I Improper Integrals, where the interval integration is semi-infinite or infinite

  Example: $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$
- Type II Improper Integrals, where the integrand is discontinuous somewhere in the interval

  Example: $\int_0^1 \ln x\,dx = \lim_{t\to 0} \int_t^1 \ln x\,dx$

The improper integral *converges* if the limit that defines it exists and is finite; otherwise it *diverges*

Gold Standard

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \textit{converges}, & p > 1 \\ \textit{diverges}, & p \leq 1 \end{cases}$$

Convergence Test for Improper Integrals

Suppose $f(x) \geq g(x) \geq 0$.

- If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges

Which Integrals Converge, and which Diverge?

1. $\displaystyle\int_1^\infty \frac{x^3}{x^5+2x+1}\,dx$
2. $\displaystyle\int_1^\infty \frac{x}{(x^2+1)^{2/3}}\,dx$
3. $\displaystyle\int_1^\infty \frac{e^{-x}}{x+1}\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^4}\,dx$