Math 114 - Series

Peter A. Perry

University of Kentucky

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Unit II: Infinite Series

Announcements

- There will be no quiz in recitation this week. Your exams will be graded on Wednesday and should be returned to you in recitation on Thursday. There will be a new version of Worksheet 9 to download for Thursday's recitation. We will not be using the original worksheets 7 and 9.
- Remember that you have Webwork A6 on sequences due this Friday, and Webwork B1 on sequences by recursion due this coming Monday.

Summation Notation

Write out $\sum_{n=1}^{4} \frac{1}{2 \cdot 3^n}$ as a sum.

A. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$
B. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$
C. $\frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162}$
D. $1 + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162}$

Zeno's Paradox: You Can't Get There from Here

Zeno of Elea (c. 490 c. 430 BC) was a Greek philosopher who is remembered for his 'paradoxes.'

Zeno's paradox about motion goes like this.

Suppose I want to walk a mile. First I have to walk 1/2 mile. Then I have to walk half the distance again, or 1/4 mile. Then I have to walk half the distance again, or 1/8 mile...–so I can *never* finish walking.

Why Zeno Was Wrong

Suppose I walk at 3 miles per hour. How much time does each part of the trip take?

1/2 mile	1/4 mile	1/8 mile	...	$1/2^n$ mile
1/6 hour	1/12 hour	1/24 hour	...	$1/(3 \cdot 2^n)$ hour

What is the total time taken?

The times t_n for each part of the trip form a *sequence* or *list of numbers*.

To find the total time taken we add up all the numbers in this list.

The sum of *all* the numbers in the list is the *series* $\sum_{n=1}^{\infty} t_n$ or

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n}$$

Why Zeno Was Wrong

How do you add up an *infinite list* of numbers?

Think of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n}$ as an "improper sum." So, maybe we should try to define it as

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{3 \cdot 2^n}.$$

The numbers

$$s_N = \sum_{n=1}^{N} \frac{1}{3 \cdot 2^n}$$

are called *partial sums*. So, the infinite sum converges if the partial sums have a limit.

A Trick for Finding Sums

We pause for a bit of algebra to help with partial sums.

What is $S_N = \sum_{n=1}^{N} ar^{n-1}$ for numbers a and r?

$$\begin{aligned}(1-r)S_N &= a + ar + \cdots ar^{N-1} \\ &\quad - \left(ar + ar^2 + \ldots + ar^{N-1} + ar^N\right) \\ &= a - ar^N\end{aligned}$$

so

$$S_N = \frac{a(1-r^N)}{1-r}$$

Find the Limit

Find $\lim_{n\to\infty}(2-(1/3)^n)$

A. -1

B. $5/3$

C. 0

D. 2

Why Zeno Was Wrong

If $S_N = \sum_{n=1}^{N} ar^{n-1}$, then $S_N = \dfrac{a(1-r^N)}{1-r}$

The total time is

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{6} \left(\frac{1}{2}\right)^{n-1}$$

The time after N steps is

$$S_N = \sum_{n=1}^{N} \frac{1}{6} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{6} \cdot \frac{1-(1/2)^N}{1-(1/2)}$$

What happens as $N \to \infty$?

Infinite Series

If $\{a_n\}$ is a *sequence*, the *infinite series* formed from that sequence is denoted $\sum_{n=1}^{\infty} a_n$ and is defined to be

$$\lim_{N\to\infty} s_N$$

where

$$s_N = \sum_{n=1}^{N} a_n$$

The numbers s_N are called the *partial sums* of the series. The series either *converges* or *diverges* depending on whether $\lim_{N\to\infty} s_N$ has a limit.

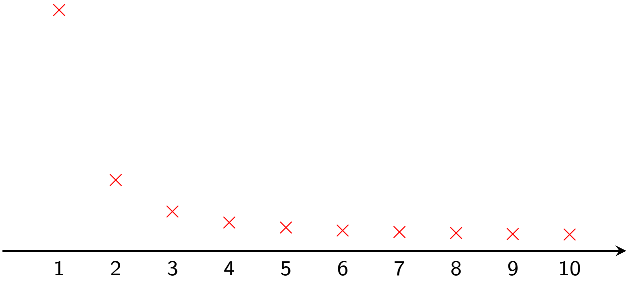
Analogy: Improper Integrals and Infinite Series

An infinite series is like an "Improper Sum"

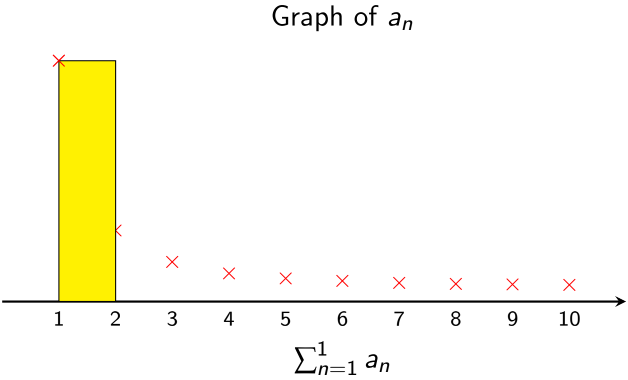
Improper Integral	**Infinite Series**
The function $f(x)$	The sequence a_n
The integral $\int_1^\infty f(x)\,dx$	The series $\sum_{n=1}^\infty a_n$
The integral $A(t) = \int_1^t f(x)\,dx$	The partial sum $s_N = \sum_{n=1}^N a_n$
$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} A(t)$	$\sum_{n=1}^\infty a_n = \lim_{N\to\infty} s_N$

A Picture of Infinite Series

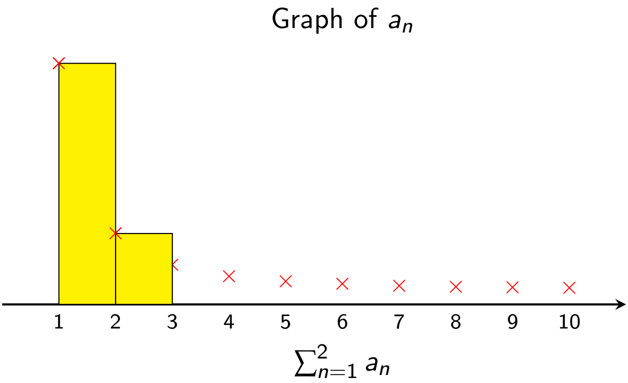
Graph of a_n



A Picture of Infinite Series



A Picture of Infinite Series



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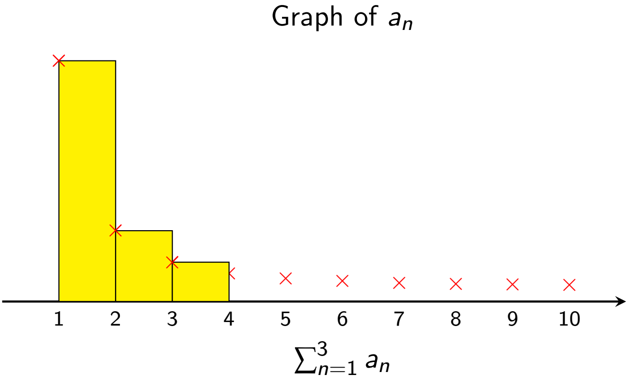

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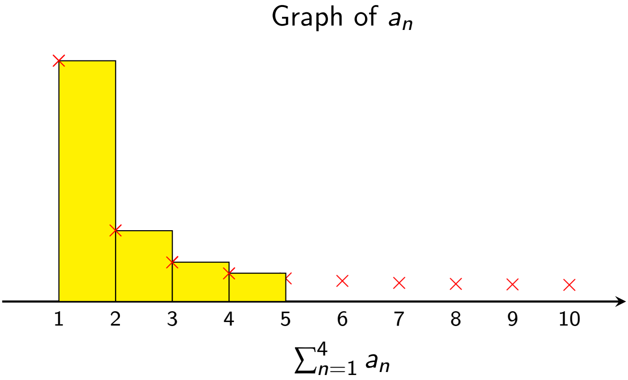
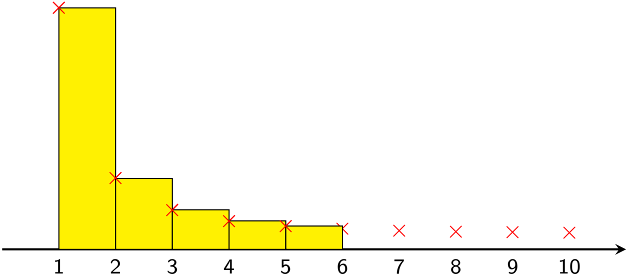


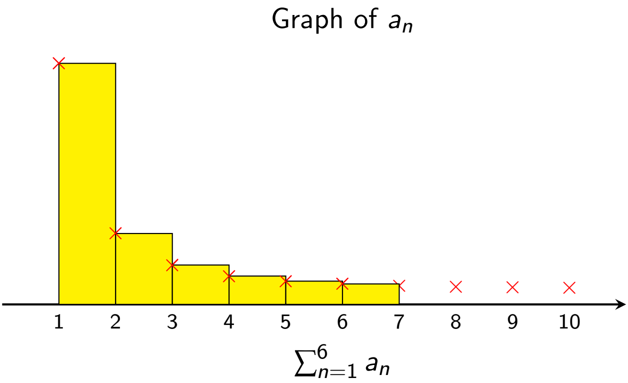
A Picture of Infinite Series

Graph of a_n



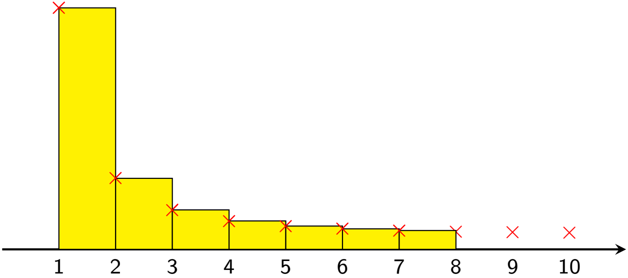
$\sum_{n=1}^{5} a_n$

A Picture of Infinite Series



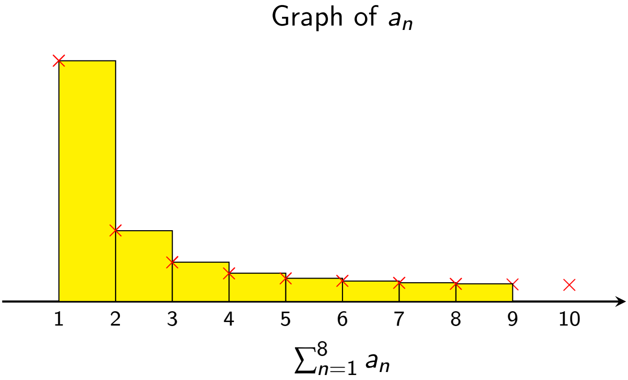
A Picture of Infinite Series

Graph of a_n



$\sum_{n=1}^{7} a_n$

A Picture of Infinite Series



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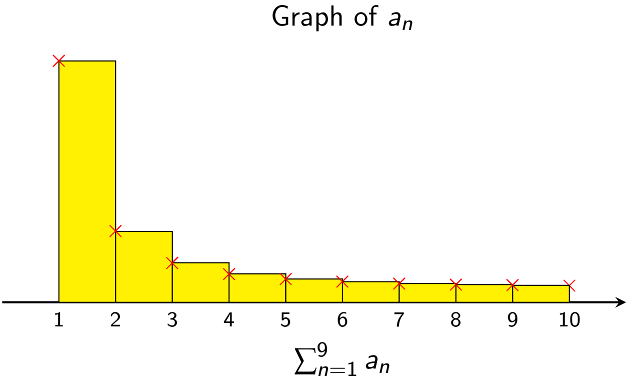

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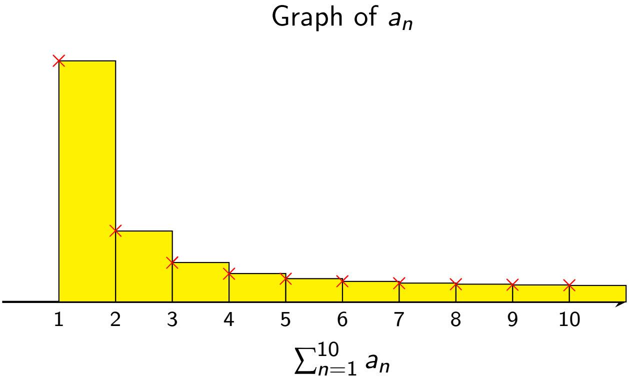

Series Puzzler

Thinking of series as the "area under the graph of a sequence," which of these series should converge?

A. $\sum_{n=1}^{\infty} 1$

B. $\sum_{n=1}^{\infty} 2^n$

C. $\sum_{n=1}^{\infty} \frac{1}{n}$

D. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Another Infinite Sum

If $S_N = \sum_{n=1}^{N} ar^{n-1}$, then $S_N = \dfrac{a(1-r^N)}{1-r}$

Does $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converge?

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \lim_{N\to\infty} s_N \text{ where } s_N = \sum_{n=1}^{N} \frac{2^n}{3^n}$$

What is s_N?

Note that $\dfrac{2^n}{3^n} = \dfrac{2}{3}\left(\dfrac{2}{3}\right)^{n-1}$. Can you now find s_N?

Yet Another Infinite Sum

Suppose x is a real number between 0 and 1. Can you find $\sum_{n=1}^{\infty} x^n$?

Remember

If $S_N = \sum_{n=1}^{N} ar^{n-1}$, then $S_N = \dfrac{a(1-r^N)}{1-r}$

Does This Series Converge?

Consider the infinite sum

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/2) + \ldots$$

What are the partial sums s_1, s_2, and s_3?

Can you guess the formula for s_N?

Does $\lim_{N \to \infty} s_N$ exist?

Does This Series Converge?

$$\sum_{n=1}^{\infty} \frac{2}{n^2 - 1}$$

Hint: Use summation by partial fractions!

Series Arithmetic

Infinite series obey the rules

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$$

Suppose that $\sum_{n=1}^{\infty} a_n = 3$ and $\sum_{n=1}^{\infty} b_n = 5$. What is $\sum_{n=1}^{\infty} 2a_n + 4b_n$?

A. 3

B. 5

C. 21

D. 26

E. 36