# Math 114 - Series, Integral Test

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# Unit II: Infinite Series


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|---|---|
| Lecture 1 | Introduction to Series |
| Lecture 2 | The Integral Test |
| Lecture 3 | The Comparison and Limit Comparison Tests |
| Lecture 4 | Alternating Series |
| Lecture 5 | Absolute and Conditional Convergence |
| Lecture 6 | The Ratio and Root Tests |
| Lecture 7 | Power Series |
| Lecture 8 | Representing Functions as Power Series |
| Lecture 9 | Taylor Series |
| Lecture 10 | Exam II Review |
| Lecture 11 | Exam II Review |

# Announcements

- Remember that Webwork A6 on sequences due tonight and Webwork B1 on sequences by recursion due this coming Monday, February 12. Webwork B2 on series is due on Wednesday February 14, and Webwork B3 on the integral test is due on Friday February 16.

# Sequence,Infinite Series, Partial Sum

Given a *sequence* (a list of numbers $\{a_n\}$), we seekto define the infinite series

$$\sum_{n=1}^{\infty} a_n$$

as a limit of *partial sums*

$$s_N = \sum_{n=1}^{N} a_n$$

That is,

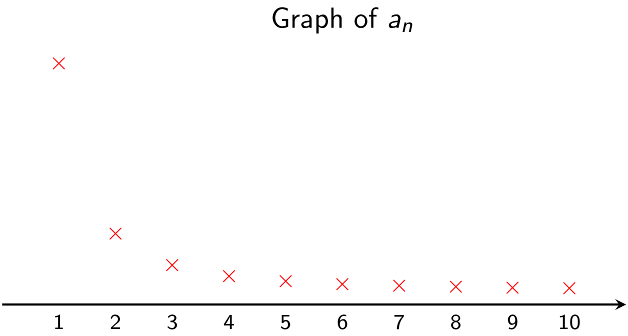
$$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{the infinite series}} = \underbrace{\lim_{N\to\infty} s_N}_{\text{the limit of partial sums}}$$

# Analogy: Improper Integrals and Infinite Series

An infinite series is like an "Improper Sum"

| **Improper Integral** | **Infinite Series** |
|---|---|
| The function $f(x)$ | The sequence $a_n$ |
| The integral $\int_1^\infty f(x)\,dx$ | The series $\sum_{n=1}^\infty a_n$ |
| The integral $A(t) = \int_1^t f(x)\,dx$ | The partial sum $s_N = \sum_{n=1}^N a_n$ |
| $\int_1^\infty f(x)\,dx = \lim_{t\to\infty} A(t)$ | $\sum_{n=1}^\infty a_n = \lim_{N\to\infty} s_N$ |

# A Picture of Infinite Series



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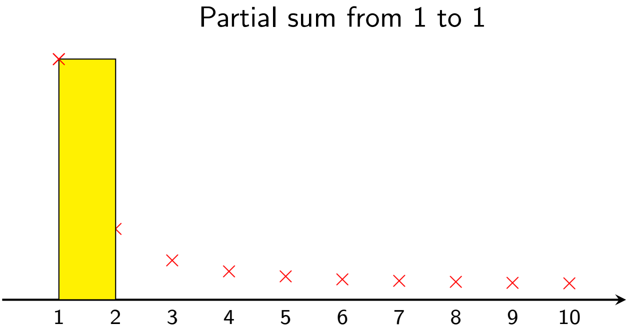
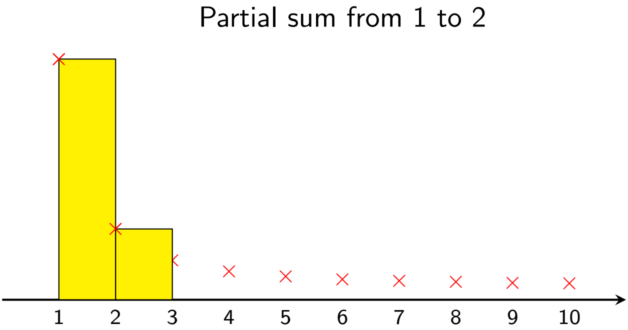


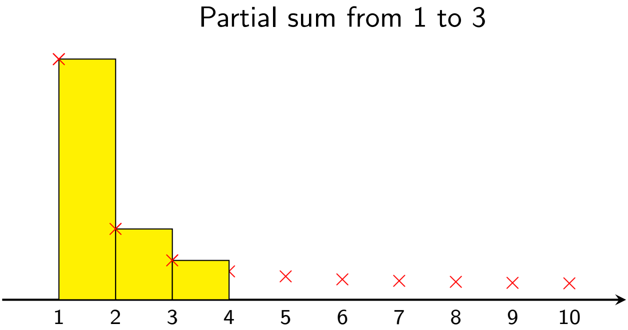

$s_1 = \sum_{n=1}^{1} a_n$

# A Picture of Infinite Series



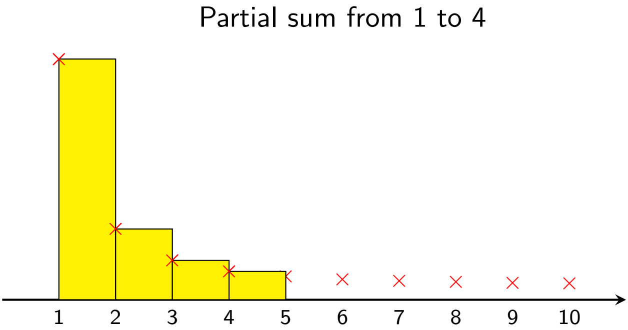

$s_2 = \sum_{n=1}^{2} a_n$

# A Picture of Infinite Series




$s_3 = \sum_{n=1}^{3} a_n$

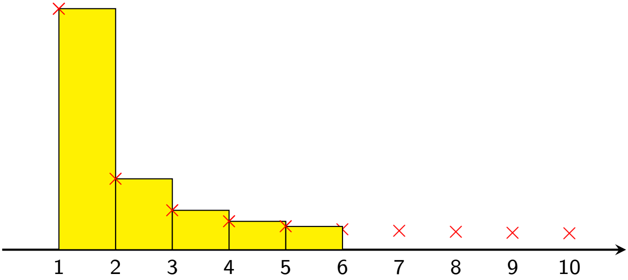
# A Picture of Infinite Series




$$s_4 = \sum_{n=1}^{4} a_n$$

# A Picture of Infinite Series

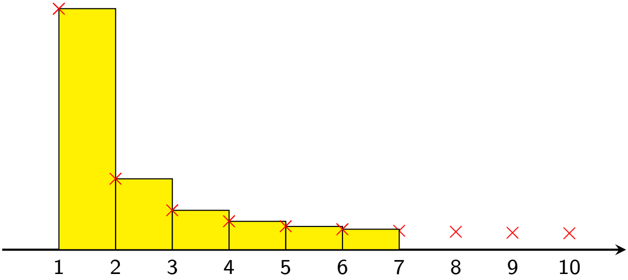
Partial sum from 1 to 5




$s_5 = \sum_{n=1}^{5} a_n$

# A Picture of Infinite Series

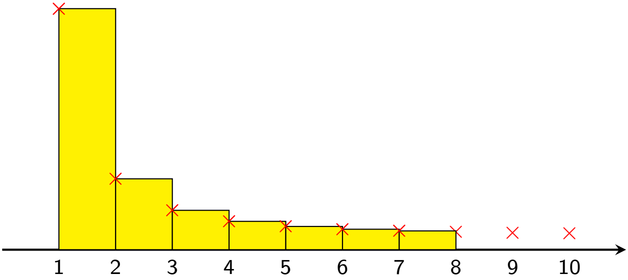
Partial sum from 1 to 6




$s_6 = \sum_{n=1}^{6} a_n$

# A Picture of Infinite Series

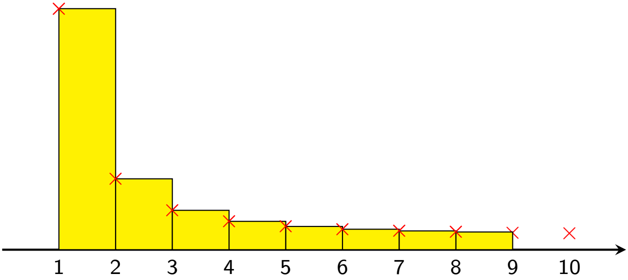
Partial sum from 1 to 7




$s_7 = \sum_{n=1}^{7} a_n$

# A Picture of Infinite Series

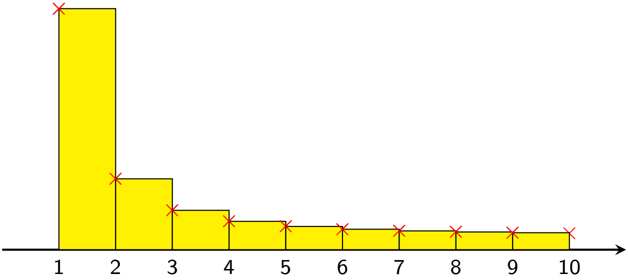
Partial sum from 1 to 8




$$s_8 = \sum_{n=1}^{8} a_n$$

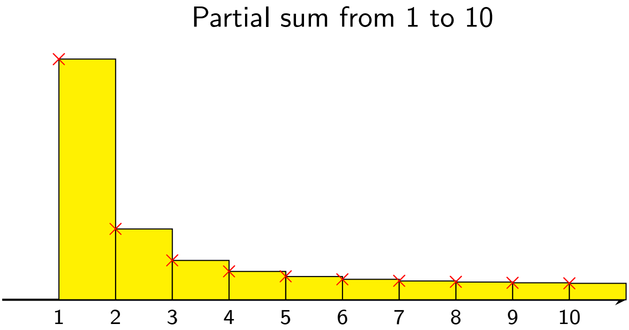
# A Picture of Infinite Series

Partial sum from 1 to 9




$s_9 = \sum_{n=1}^{9} a_n$

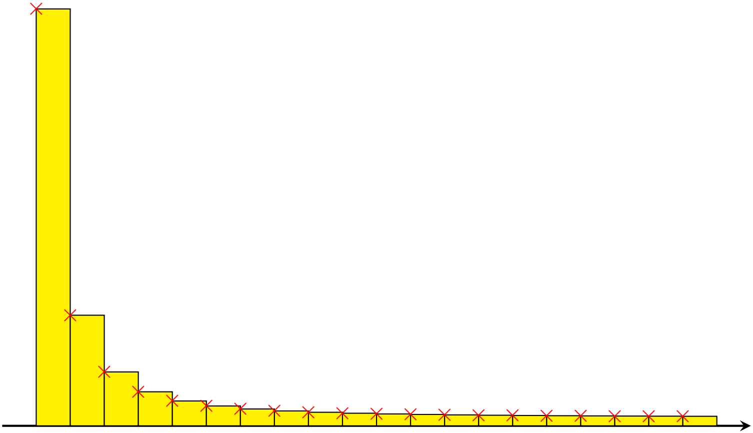
# A Picture of Infinite Series




$$s_{10} = \sum_{n=1}^{10} a_n$$

# A Picture of Infinite Series

The infinite sum $\sum_{n=1}^{\infty} a_n$ converges if and only if the *total* area of *all* the rectangles is finite!



# Yet Another Infinite Sum

Suppose $x$ is a real number between 0 and 1. Can you find $\sum_{n=1}^{\infty} x^n$?

Remember

If $S_N = \sum_{n=1}^{N} ar^{n-1}$, then $S_N = \dfrac{a(1-r^N)}{1-r}$

Solution: The series $\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \dots$ is a geometric series with $a = x$, $r = x$. From the above formula

$$\begin{aligned} S_N &= \frac{x(1-x^N)}{1-x} \\ \lim_{N\to\infty} S_N &= \frac{x}{1-x} \lim_{x\to\infty} (1-x^N) \\ &= \frac{x}{1-x}. \end{aligned}$$

# Does This Series Converge?

Consider the infinite sum

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=(1-1/2)+(1/2-1/3)+(1/3-1/4)+\ldots$$

Find the partial sum $s_4$ for this series.

A. 3/4
B. 4/5
C. 5/6
D. 6/7
E. 1/4

# Does This Series Converge?

Consider the infinite sum

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=(1-1/2)+(1/2-1/3)+(1/3-1/2)+\dots$$

Can you guess the formula for $s_N$?

A. $\frac{1}{2}\frac{1-(1/2)^N}{1/2}$
B. $\ln N$
C. $\frac{N-1}{N}$
D. $\frac{N}{N+1}$

Does $\lim_{N\to\infty} s_N$ exist? Yes. $sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1.$

# Series Arithmetic

Infinite series obey the rules

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$$

Suppose that $\sum_{n=1}^{\infty} a_n = 3$ and $\sum_{n=1}^{\infty} b_n = 5$. What is $\sum_{n=1}^{\infty} 2a_n + 4b_n$?

A. 3

B. 5

C. 21

D. 26

E. 36

# Why Does $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverge?

Recall that

$$\int_1^N \frac{1}{x}\,dx = \ln N$$

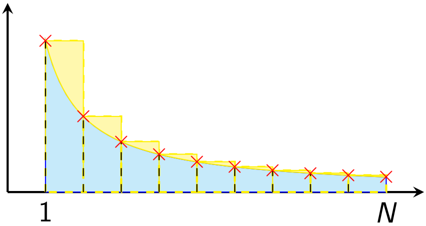
"By picture"

$$\sum_{n=1}^{N} \frac{1}{n} > \int_1^N \frac{1}{x}\,dx$$

So

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} > \lim_{n\to\infty} \ln N = +\infty$$

so this series, the *harmonic series*, must *diverge*



# Why does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converge?

Recall that

$$\int_1^N \frac{1}{x^2}\,dx = 1 - \frac{1}{N}$$

"By Picture"

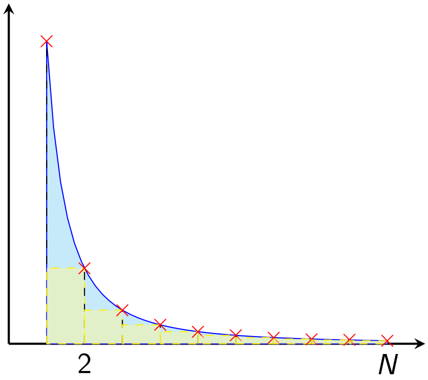
$$\sum_{n=2}^{N} \frac{1}{n^2} < \int_1^N \frac{1}{x^2}\,dx$$

So

$$\lim_{N\to\infty} \sum_{n=2}^{\infty} \frac{1}{n^2} < 1$$

So this series must *converge*.
If you take some more mathematics, you'll learn how to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$



# What Do These Examples Have in Common?

$\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges because $\int_1^{\infty} \frac{1}{x}\,dx$ diverges

$\sum_{n=1}^{\infty} \dfrac{1}{x^2}$ conveges because $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

In each case, $a_n = f(n)$ where $f$ is a positive, decreasing, continuous function. In the first case $f(x) = 1/x$, and in the second case, $f(x) = 1/x^2$.

# Series Puzzler

For what values of $p$ does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

A. $p < 1$

B. $p \leq 1$

C. $p > 1$

D. $p \geq 1$

E. All values of $p$

# The Integral Test

Suppose that $a_n = f(n)$ where $f$ is a *positive*, *continuous*, *decreasing* function.

- If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges
- If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges

# Using the Integral Test

Suppose that $a_n = f(n)$ where $f$ is a *positive*, *continuous*, *decreasing* function.

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- If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges

Determine whether each of the following series converges or diverges.

- $\sum_{n=1}^\infty 2ne^{-n^2}$ (so $f(x) = 2xe^{-x^2}$) converges
- $\sum_{n=1}^\infty \frac{1}{1+n^2}$ (so $f(x) = \frac{1}{1+x^2}$) converges
- $\sum_{n=2}^\infty \frac{1}{n \ln n}$ (so $f(x) = \frac{1}{x \ln x}$ diverges
- $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ (so $f(x) = \frac{1}{x(\ln x)^2}$) converges

# What is a Remainder?

If a series $\sum_{n=1}^{\infty} a_n$ converges to a value $s$, the *remainder* $R_N$ is the difference between the $N$th partial sum and the true value:

$$\underbrace{s}_{\text{true value}} = \underbrace{s_N}_{N\text{th partial sum}} + \underbrace{R_N}_{\textit{remainder}}$$

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**Example** For the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, the true value $s$ is $\pi^2/6 \simeq 1.64493$, and $s_{10} = 1.54977$. So the *remainder* is (to five decimal places) 0.09516

$$\underbrace{\frac{\pi^2}{6}}_{\text{true value}} \simeq \underbrace{1.54977}_{\text{10th partial sum}} + \underbrace{0.09516}_{\text{remainder } R_{10}}$$

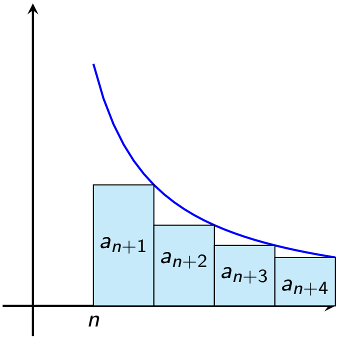
# How Do You Estimate the Remainder?

Remember that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

If $f$ is positive, decreasing, and continuous, $a_k = f(k)$, $s = \sum_{n=1}^{\infty} a_k$, then

$$R_n < \int_n^{\infty} f(x)\, dx$$



# How Do You Estimate the Remainder?

Remember that

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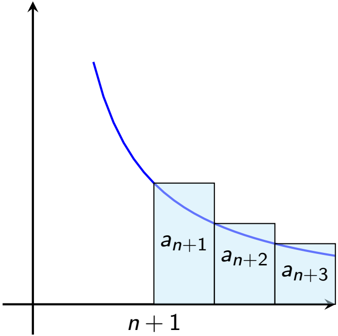
If $f$ is positive, decreasing, and continuous, $a_k = f(k)$, $s = \sum_{n=1}^{\infty} a_k$, then

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$



# Remainder Estimates

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

If $f$ is positive, decreasing, and continuous, $a_k = f(k)$, $s = \sum_{n=1}^{\infty} a_k$, then

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

Leonhard Euler (1707-1783) determined that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

The 10th partial sum for this series is about 1.0820365. How big is the remainder?

The function $f(x) = 1/x^4$ and $n = 10$.

# Remainder Estimates

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

If $f$ is positive, decreasing, and continuous, $a_k = f(k)$, $s = \sum_{n=1}^{\infty} a_k$, then

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The 10th partial sum for this series is about 1.0820365. How big is the remainder?

The function $f(x) = 1/x^4$ and $n = 10$.

$$\int_{11}^{\infty} \frac{1}{x^4}\,dx = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{(11)^3} - \frac{1}{n^3}\right) \simeq 0.00025044$$

$$\int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{10^3} - \frac{1}{n^3}\right) \simeq 0.00033333$$