Math 114 - The Comparison Test

Peter A. Perry

University of Kentucky

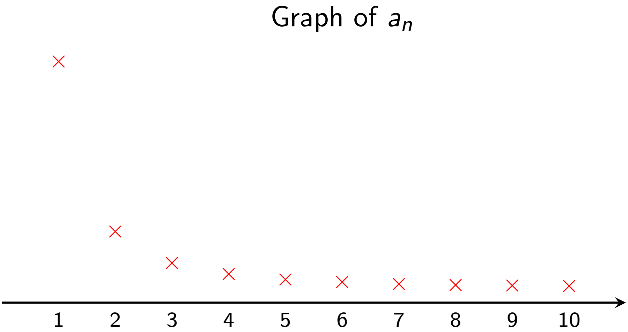
February 12, 2018

Unit II: Infinite Series

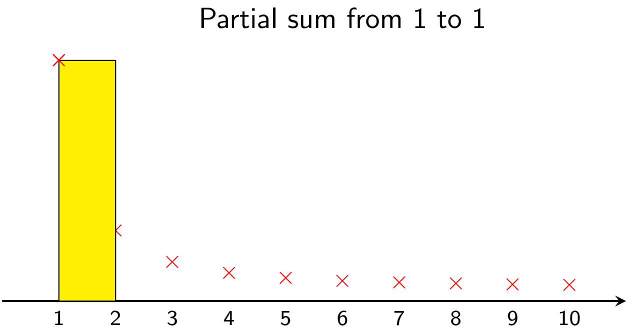
Announcements

- We are actually one day *ahead* of the course calendar, which will give us room to slow down later if needed without jeopardizing our two review days before exam II. Please read ahead in the text (11.5 for Wednesday, 11.6 for Friday)
- Remember Webwork B1 on sequences by recursion is due tonight, Monday, February 12. Webwork B2 on series is due on Wednesday February 14, and Webwork B3 on the integral test is due on Friday February 16.
- In recitation on Tuesday February 13, you'll work on the series and the integral test. In recitation on Thursday, February 15, you'll work on the Comparison and Limit Comparison tests for convergence. You'll have a quiz, Quiz 4, on sections 11.1–11.2 – basics of sequences and series.

A Picture of Infinite Series



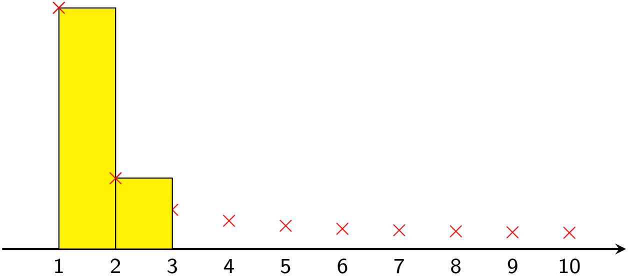
A Picture of Infinite Series



$s_1 = \sum_{n=1}^{1} a_n$

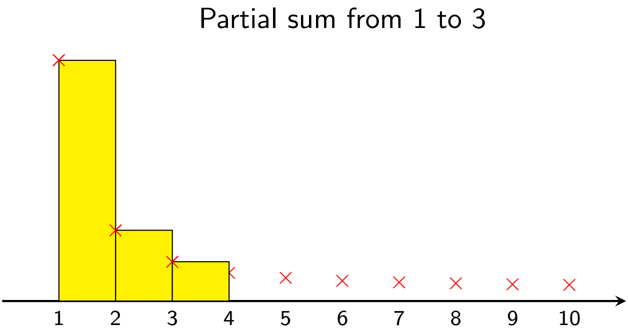
A Picture of Infinite Series

Partial sum from 1 to 2



$s_2 = \sum_{n=1}^{2} a_n$

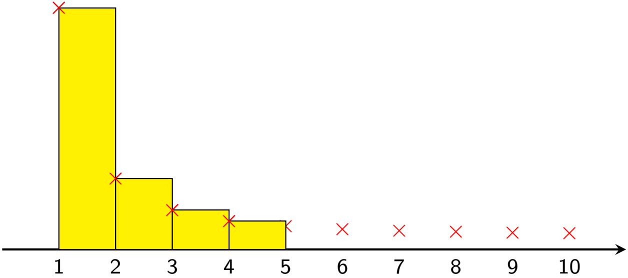
A Picture of Infinite Series



$s_3 = \sum_{n=1}^{3} a_n$

A Picture of Infinite Series

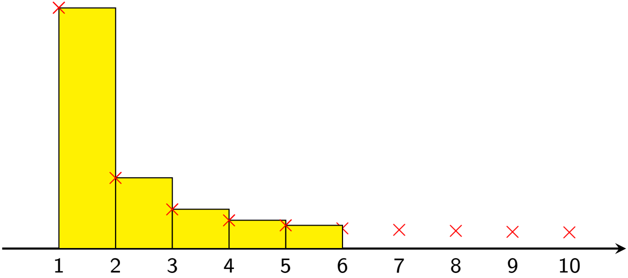
Partial sum from 1 to 4



$s_4 = \sum_{n=1}^{4} a_n$

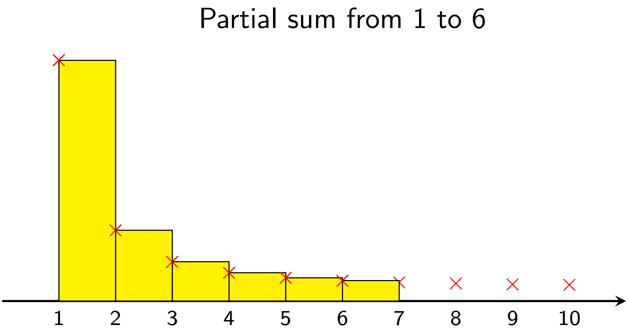
A Picture of Infinite Series

Partial sum from 1 to 5



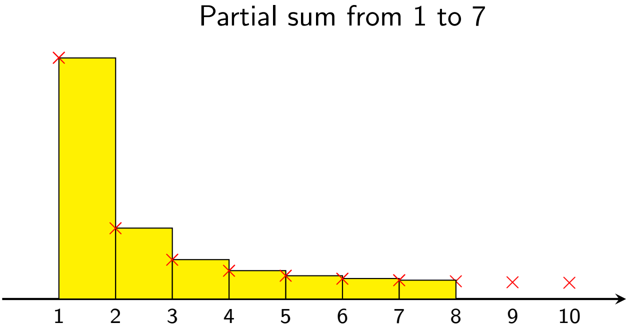
$s_5 = \sum_{n=1}^{5} a_n$

A Picture of Infinite Series



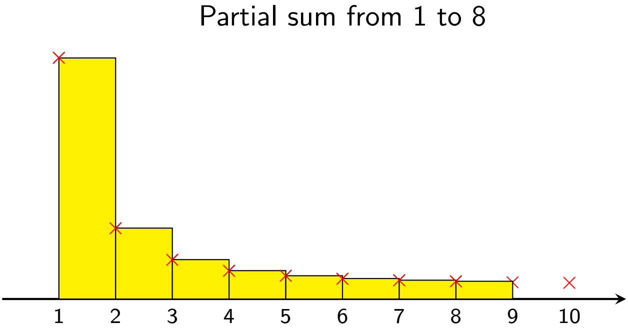
$s_6 = \sum_{n=1}^{6} a_n$

A Picture of Infinite Series



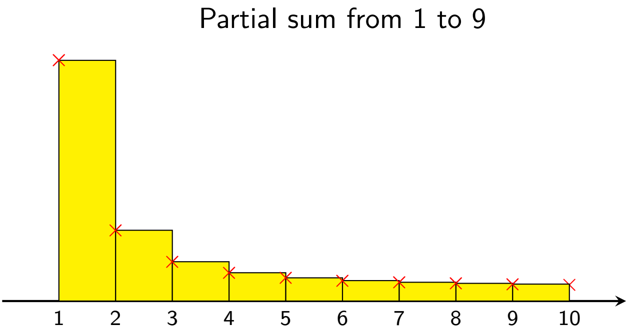
$s_7 = \sum_{n=1}^{7} a_n$

A Picture of Infinite Series



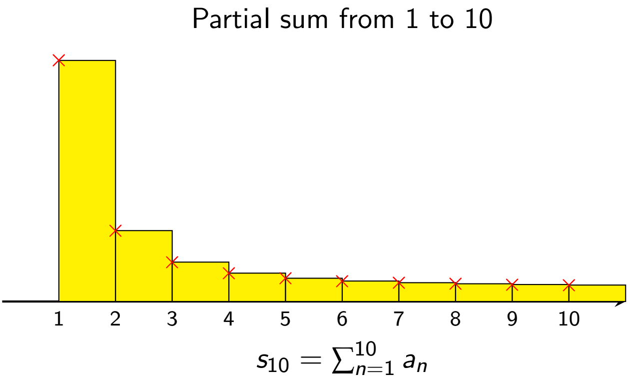
$s_8 = \sum_{n=1}^{8} a_n$

A Picture of Infinite Series



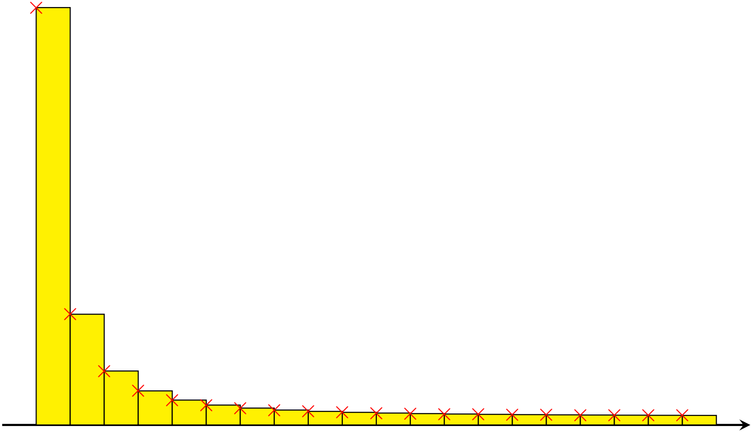
$s_9 = \sum_{n=1}^{9} a_n$

A Picture of Infinite Series



A Picture of Infinite Series

The infinite sum $\sum_{n=1}^{\infty} a_n$ converges if and only if the *total* area of *all* the rectangles is finite!



Recap

We've said that an *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* if the partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

approach a limit s as $N \to \infty$.

- If $\lim_{N\to\infty} s_n = s$ exists, then the series *converges* and $\sum_{n=1}^{\infty} a_n = s$
- If $\lim_{N\to\infty} s_n$ is infinite or does not exist, the series *diverges*

Two Statements about Convergence and Divergence

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
- If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Which of the following series converges?

A. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

B. $\sum_{n=1}^{\infty} (-1)^n$

C. $\sum_{n=1}^{\infty} \frac{1}{n}$

D. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Two Statements about Convergence and Divergence

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
- If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

On the other hand, even if $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ can still diverge! Can you think of an example?

The condition $\lim_{n\to\infty} a_n = 0$ is a convergence condition that mathematicians call "necessary but not sufficient."

Recap

We saw two kinds of series where you can actually compute the partial sums:

- Geometric series: The Nth partial sum for $\sum_{n=1}^{\infty} ar^{n-1}$ (a is the first term, r is the ratio of successive terms) is

$$s_N = a\frac{(1-r^N)}{1-r}$$

  The partial sums have a limit as $N \to \infty$ provided $|r| < 1$

- Telescoping series: For series like

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

  all of the "middle terms" cancel and you can compute

$$s_N = 1 - \frac{1}{N+1}$$

Unhappily, for most series, we can't compute s_N explicitly. This is why we need *convergence tests*.

Integral Test Review

Suppose $a_n = f(n)$ where f is a positive, decreasing, continuous function. The sum $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Suppose $a_n = f(n)$ where f is a positive, decreasing, continuous function, and suppose that $\sum_{n=1}^{\infty} a_n$ converges to a number s. We can estimate the *remainder* $R_n = s - s_N$ by

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

How many terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ should be summed to give an error of no more than 0.001?

A New Gold Standard

Remember that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges,} & p > 1 \\ \text{diverges,} & p \leq 1. \end{cases}$$

Since $f(x) = x^{-p}$ is *positive, decreasing, and continuous*, we now know that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges,} & p > 1 \\ \text{diverges,} & p \leq 1. \end{cases}$$

The series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is called a p-series. If $p = 1$ the series is called the *harmonic series*.

A Guide to Convergence Tests

- Integral test for $\sum_{n=1}^{\infty} a_n$ if $a_n = f(n)$ for a positive, continuous decreasing function f (Lecture 2)
- Comparison test and limit comparison test for positive series (Lecture 3)
- Alternating series tests for 'alternating' series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ (Lecture 4)
- Ratio and Root Tests for absolute convergence of series (Lecture 5)

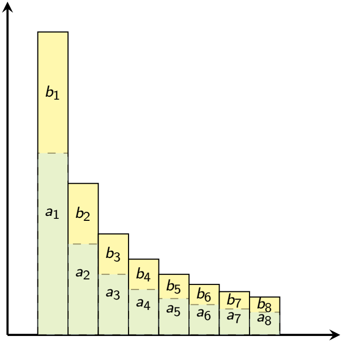
Comparison for Convergence

Suppose that $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ for all n. What does this imply about the series $\sum_{n=1}^{\infty} a_n$?

A. The series $\sum_{n=1}^{\infty} a_n$ diverges

B. The series $\sum_{n=1}^{\infty} a_n$ converges

C. The series $\sum_{n=1}^{\infty} a_n$ could converge or diverge.



Comparison for Convergence: Example

Suppose that $b_n = \dfrac{1}{n^2}$ and $a_n = \dfrac{1}{n^2 + 5}$.

Comparison for Convergence: Example

Suppose that $b_n = \dfrac{1}{n^2}$ and $a_n = \dfrac{1}{n^2 + 5}$.

Why is it true that $0 \leq a_n \leq b_n$?

Comparison for Convergence: Example

Suppose that $b_n = \dfrac{1}{n^2}$ and $a_n = \dfrac{1}{n^2+5}$.

Why is it true that $0 \leq a_n \leq b_n$?

Why does $\sum_{n=1}^{\infty} b_n$ converge?

Comparison for Convergence: Example

Suppose that $b_n = \dfrac{1}{n^2}$ and $a_n = \dfrac{1}{n^2+5}$.

Why is it true that $0 \leq a_n \leq b_n$?

Why does $\sum_{n=1}^{\infty} b_n$ converge?

What can we conclude about $\sum_{n=1}^{\infty} a_n$?

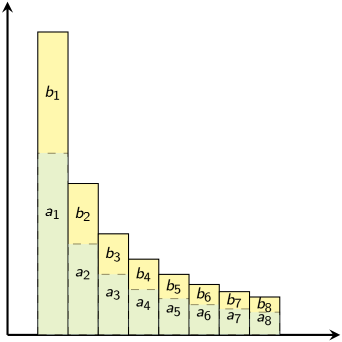
Comparison for Divergence

Suppose that $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$ for all n. What does this imply about the series $\sum_{n=1}^{\infty} b_n$?

A. The series $\sum_{n=1}^{\infty} a_n$ diverges

B. The series $\sum_{n=1}^{\infty} a_n$ converges

C. The series $\sum_{n=1}^{\infty} a_n$ could converge or diverge.



Comparison for Divergence: Example

Suppose that $b_n = \dfrac{2 + \cos(1/n)}{n}$ and $a_n = \dfrac{1}{n}$.

Comparison for Divergence: Example

Suppose that $b_n = \dfrac{2 + \cos(1/n)}{n}$ and $a_n = \dfrac{1}{n}$.

Why is it true that $0 \leq a_n \leq b_n$?

Comparison for Divergence: Example

Suppose that $b_n = \dfrac{2 + \cos(1/n)}{n}$ and $a_n = \dfrac{1}{n}$.

Why is it true that $0 \leq a_n \leq b_n$?

Why does $\sum_{n=1}^{\infty} b_n$ diverge?

Comparison for Divergence: Example

Suppose that $b_n = \dfrac{2 + \cos(1/n)}{n}$ and $a_n = \dfrac{1}{n}$.

Why is it true that $0 \leq a_n \leq b_n$?

Why does $\sum_{n=1}^{\infty} b_n$ diverge?

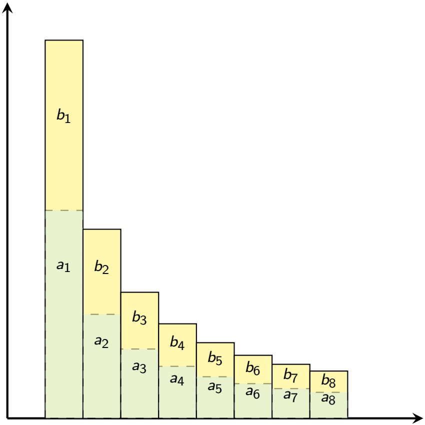
What can we conclude about $\sum_{n=1}^{\infty} a_n$?

The Comparison Test

The Comparison Test

Suppose that $0 \leq a_n \leq b_n$.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.



The Limit Comparison Test

Sometimes, it can be hard to prove that $0 \leq a_n \leq b_n$. An easier test to apply is the *limit comparison test*.

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

The Limit Comparison Test

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

The limit condition implies that if n is large enough (say $n > N$)

$$m < \frac{a_n}{b_n} < M$$

for positive numbers m and M so

$$mb_n < a_n < Mb_n$$

Now you can use the Comparison test.

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

Example Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4n+2}{n^3+18n+46}$ converge or diverges?

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

Example Does the series $\sum_{n=1}^{\infty} \frac{n^2+4n+2}{n^3+18n+46}$ converge or diverges?

Look at the leading powers in numerator and denominator:

$$a_n = \frac{n^{2}+4n+2}{n^{3}+18n+46}$$

Guess that this sequence behaves like the 'gold standard' sequence $b_n = \frac{1}{n}$

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

Example Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4n+2}{n^3+18n+46}$ converge or diverges?

Look at the leading powers in numerator and denominator:

$$a_n = \frac{n^2+4n+2}{n^3+18n+46}$$

Guess that this sequence behaves like the 'gold standard' sequence $b_n = \dfrac{1}{n}$

Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+4n+2}{n^3+18n+46} \cdot n = 1$$

and conclude that the series *diverges*.

The Limit Comparison Test

Limit Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite, positive number, then either both series converge or both series diverge.

Does the series $\sum \frac{n^2+n+1}{\sqrt{n^7+2n^2+4}}$ converge or diverge?

A. Converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^5}$

B. Converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

C. Diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

D. Diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$