Math 114 - The Alternating Series Test

Peter A. Perry

University of Kentucky

February 14, 2018


PyeongChang 2018
1

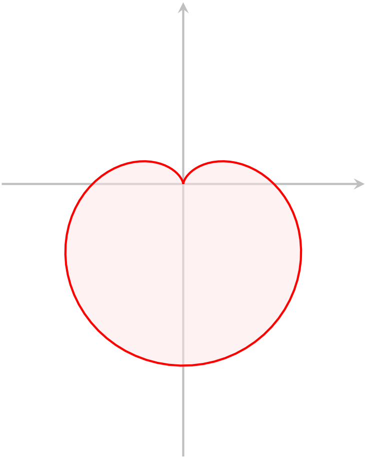

PyeongChang 2018
2

Happy Cardioid Day!

The Cardoid Curve is given by the polar equation

$$r = (1 - \sin\theta)$$

We'll study polar curves in Unit III.



Unit II: Infinite Series

Announcements

- Remember Webwork B2 on series is due on Wednesday February 14, and Webwork B3 on the integral test is due on Friday February 16.

Recap

We've said that an *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* if the partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

approach a limit s as $N \to \infty$.

- If $\lim_{N\to\infty} s_n = s$ exists, then the series *converges* and $\sum_{n=1}^{\infty} a_n = s$
- If $\lim_{N\to\infty} s_n$ is infinite or does not exist, the series *diverges*

Unhappily, for most series, we can't compute s_N explicitly. This is why we need *convergence tests*.

Which is Which?

A. A *sequence* is the sum of an infinite ordered list of numbers, and a *series* is an ordered list of real numbers, in one-to-one correspondence with the positive integers $1, 2, \ldots$.

B. A *series* is the sum of an infinite ordered list of real numbers, and a *sequence* is an ordered list of real numbers in one-to-one correspondence with the positive integers $1, 2, \ldots$

C. None of the Above

A Guide to Convergence Tests

- Divergent Series Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges
- Integral test for $\sum_{n=1}^{\infty} a_n$ if $a_n = f(n)$ for a positive, continuous decreasing function f (Lecture 2)
- Comparison test and limit comparison test for positive series (Lecture 3)
- Alternating series tests for 'alternating' series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ (Lecture 4)
- Ratio and Root Tests for absolute convergence of series (Lecture 5)

Which Test Would You Use?

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$$

A. Integral Test
B. Comparison Test
C. Limit Comparison Test
D. None of the above

$$\sum_{n=1}^{\infty} \frac{n+1}{3n^3+n^2+1}$$

A. Integral Test
B. Comparison Test
C. Limit Comparison Test
D. None of the above

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

A. Integral Test
B. Comparison Test
C. Limit Comparison Test
D. None of the above

A New Kind of Series

Remember that the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (why?).

What about the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

None of the previous tests are applicable because the terms *alternate* between positive and negative sign.

Let's look at the partial sums!

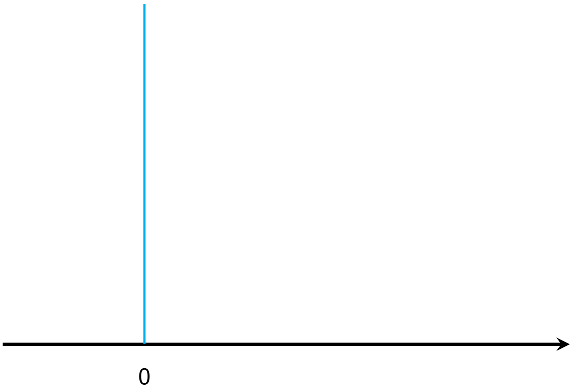
Partial sums of $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

n	s_n	n	s_n
1	1.000000	11	0.736544
2	0.500000	12	0.653211
3	0.833333	13	0.730134
4	0.583333	14	0.658705
5	0.783333	15	0.725372
6	0.616667	16	0.662872
7	0.759524	17	0.721695
8	0.635524	18	0.666240
9	0.745635	19	0.718771
10	0.645635	20	0.668771

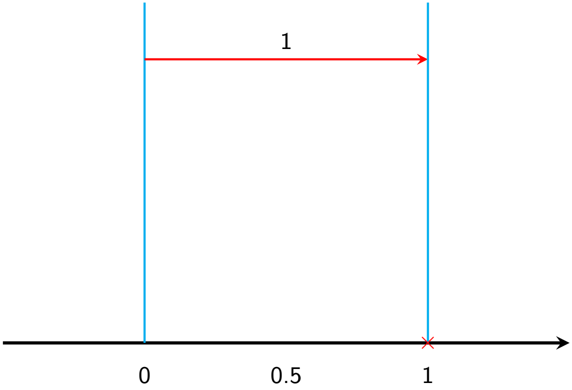
Partial sums of $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

n	s_n	n	s_n
1	1.000000	11	0.736544
2	0.500000	12	0.653211
3	0.833333	13	0.730134
4	0.583333	14	0.658705
5	0.783333	15	0.725372
6	0.616667	16	0.662872
7	0.759524	17	0.721695
8	0.635524	18	0.666240
9	0.745635	19	0.718771
10	0.645635	20	0.668771

Partial Sum Ping-Pong

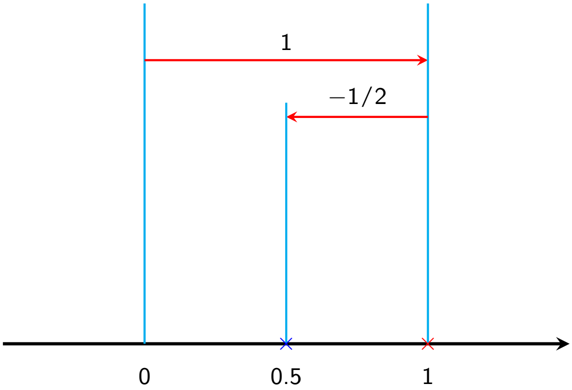


Partial Sum Ping-Pong



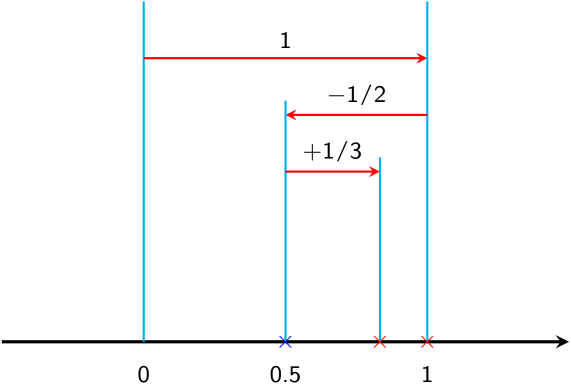
$$s_1 = 1$$

Partial Sum Ping-Pong



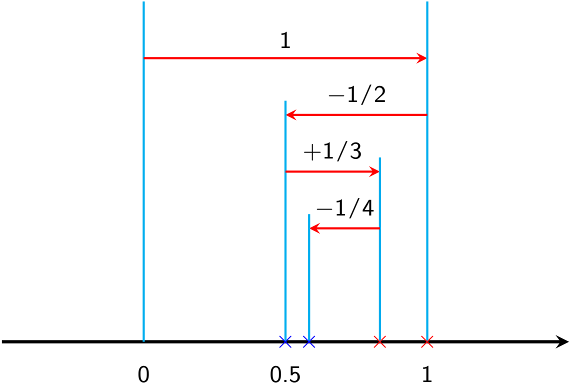
$$s_2 = 1 - \frac{1}{2}$$

Partial Sum Ping-Pong



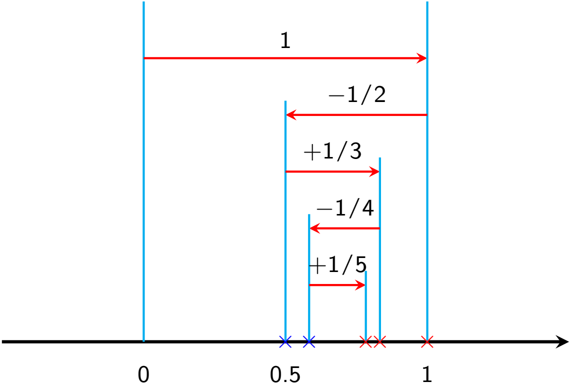
$$s_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

Partial Sum Ping-Pong



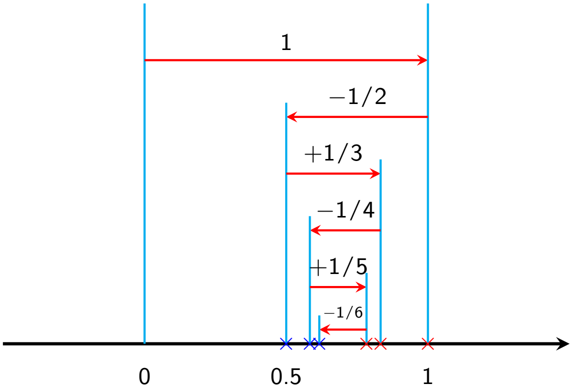
$$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

Partial Sum Ping-Pong



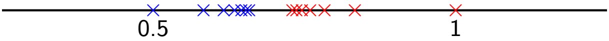
$$s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

Partial Sum Ping-Pong



$$s_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$

Partial Sum Ping-Pong



The odd partial sums $s_1, s_3, s_5, \ldots$ are *decreasing* and the even partial sums $s_2, s_4, s_6, \ldots$ are *increasing* toward a common limit!

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

The key properties of $b_n = 1/n$ are

- The sequence $\{b_n\}$ is *decreasing*
- $\lim_{n\to\infty} b_n = 0$

This means that

- The next correction is always less in size than the previous one
- The size of the corrections goes to zero

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots, \qquad b_n \geq 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Why the Alternating Series Test Works

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots, \qquad b_n \geq 0$$

The *even* partial sums are *increasing*:

$$s_{2n} = (b_1 - b_2) + (b_3 - b_4) + \ldots + (b_{2n-1} - b_{2n})$$

The *odd* partial sums are *decreasing*:

$$s_{2n+1} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots - (b_{2n} - b_{2n+1})$$

The *even* partial sums are bounded above by b_1:

$$s_{2n} = b_1 - (b_2 - b_3) - \ldots (b_{2n-2} - b_{2n-1}) - b_{2n}$$

The *even* partial sums are *increasing*:

$$s_{2n} = (b_1 - b_2) + (b_3 - b_4) + \ldots + (b_{2n-1} - b_{2n})$$

The *even* partial sums are bounded above by b_1:

$$s_{2n} = b_1 - (b_2 - b_3) - \ldots (b_{2n-2} - b_{2n-1}) - b_{2n}$$

By the *monotone sequence theorem*, $\lim_{n\to\infty} s_{2n} = s$ exists.

Since $s_{2n+1} = s_{2n} + b_{2n+1}$,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s$$

Alternating Series Test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies

(i) $0 \leq b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Does the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n+1}}$$

converge?

A. Yes

B. No

Alternating Series Test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies

(i) $0 \le b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Does the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n}$$

converge?

A. Yes

B. No

Alternating Series Test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies

(i) $0 \leq b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Does the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n+3}$$

converge?

A. Yes

B. No

If $\sum_{n=1}^{\infty} a_n$ converges to s, the *remainder* R_n is the difference between s_n and s:

$$s = s_n + R_n$$

For an alternating series, $a_n = (-1)^{n-1} b_n$ and the series converges if $\{b_n\}$ is a decreasing, positive, and $\lim_{n\to\infty} b_n = 0$.

For alternating series,

$$\boxed{|R_n| \leq b_n}$$

How many terms of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{n}$ must be added to obtain an error of no more than 0.005?

Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 1: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent, but not absolutely convergent. Such a series is called *conditionally convergent*.

Example 2: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is both convergent and absolutely convergent.

Roughly speaking a series is absolutely convergent if the convergence isn't due to cancellations between positive and negative terms.

Absolute Convergence

Theorem If a series $\sum a_n$ is absolutely convergent, it is also convergent.

Proof.

Absolute Convergence

Theorem If a series $\sum a_n$ is absolutely convergent, it is also convergent.

Proof. First, note that

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$.

Absolute Convergence

Theorem If a series $\sum a_n$ is absolutely convergent, it is also convergent.

Proof. First, note that

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$. Second, by the comparison test, $\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

Absolute Convergence

Theorem If a series $\sum a_n$ is absolutely convergent, it is also convergent.

Proof. First, note that

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$. Second, by the comparison test, $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. Since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

we see that $\sum_{n=1}^{\infty} a_n$ converges.

The Ratio Test

Remember that the "gold standard" series

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges,} & |r| < 1 \\ \text{diverges,} & |r| \geq 1 \end{cases}$$

The *ratio test* for convergence of a series $\sum_{n=1}^{\infty} a_n$ focusses on the number

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ behaves like a geometric series with "ratio" less than one in absolute value (converges)

If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ behaves like a geometric series with "ratio" larger than one in absolute value (diverges)

The Ratio Test

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive

The ratio test is inconclusive for $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ since $L = 1$!!

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If

$$\begin{cases} L < 1 & \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \\ L > 1 & \sum_{n=1}^{\infty} a_n \text{ diverges} \end{cases}$$

Does the series

$$\sum_{n=1}^{\infty} \frac{n}{5^n}$$

converge or diverge?

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If

$$\begin{cases} L < 1 & \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \\ L > 1 & \sum_{n=1}^{\infty} a_n \text{ diverges} \end{cases}$$

Does the series

$$\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$$

converge or diverge?

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. If

$$\begin{cases} L < 1 & \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \\ L > 1 & \sum_{n=1}^{\infty} a_n \text{ diverges} \end{cases}$$

Does the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$$

converge or diverge?

The Root Test

Another way of looking at our "gold standard" geometric series is that the terms $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$ behave roughly like $a_n \sim r^{n-1}$. If you take nth roots

$$(a_n)^{1/n} = a^{1/n} \left(r^{n-1}\right)^{1/n}$$

what limit do you get as $n \to \infty$?

In the *root test* for convergence of $\sum_{n=1}^{\infty} a_n$, we consider the limit

$$L = \lim_{n \to \infty} \left(|a_n|^{1/n} \right)$$

if it exists.

The Root Test

Another way of looking at our "gold standard" geometric series is that the terms $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$ behave roughly like $a_n \sim r^{n-1}$. If you take nth roots

$$\left(a_n\right)^{1/n} = a^{1/n}\left(r^{n-1}\right)^{1/n}$$

what limit do you get as $n \to \infty$?

$\lim_{n\to\infty}\left(a_n\right)^{1/n} = r$

In the *root test* for convergence of $\sum_{n=1}^{\infty} a_n$, we consider the limit

$$L = \lim_{n\to\infty}\left(\left|a_n\right|^{1/n}\right)$$

if it exists.

The Root Test

(i) If $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then the series $\sum a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$, then the series $\sum a_n$ is divergent

(iii) If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, the root test is inconclusive

Legal Notice: $|a_n|^{1/n}$ and $\sqrt[n]{|a_n|}$ are two names for the same thing

The Root Test

(i) If $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then the series $\sum a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$, then the series $\sum a_n$ is divergent

(iii) If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, the root test is inconclusive

What happens when you apply the root test to a p-series?