# Math 114 - Absolute and Conditional Convergence

Peter A. Perry

University of Kentucky

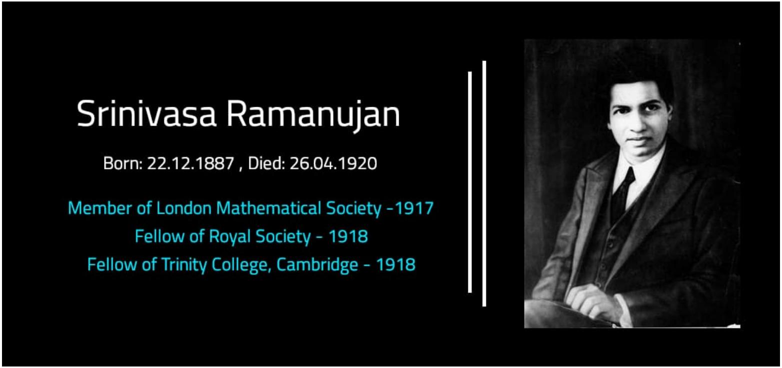
February 16, 2018

# Unit II: Infinite Series


| | |
|---|---|
| Lecture 1 | Introduction to Series |
| Lecture 2 | The Integral Test |
| Lecture 3 | The Comparison and Limit Comparison Tests |
| Lecture 4 | The Alternating Series Test |
| Lecture 5 | Absolute and Conditional Convergence |
| Lecture 6 | The Ratio and Root Tests |
| Lecture 7 | Power Series |
| Lecture 8 | Representing Functions as Power Series |
| Lecture 9 | Taylor Series |
| Lecture 10 | Exam II Review |
| Lecture 11 | Exam II Review |

# Announcements

- Webwork B3 on the integral test is due tonight!
- In next Tuesday's recitation you will work on the *Alternating Series Test* (Wednesday's lecture) and *Absolute versus Conditional Convergence* (Today's lecture)
- In next Thursday's recitation you will work on the *Ratio and Root tests* (Monday's lecture) and have a quiz on sections 11.3-11.5 (integral test, comparison test, limit comparison test, alternating series test)


Srinivasa Ramanujan
Born: 22.12.1887 , Died: 26.04.1920
Member of London Mathematical Society -1917
Fellow of Royal Society - 1918
Fellow of Trinity College, Cambridge - 1918

# Recap

We've said that an *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* if the partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

approach a limit $s$ as $N \to \infty$.

- If $\lim_{N\to\infty} s_N = s$ exists, then the series *converges* and $\sum_{n=1}^{\infty} a_n = s$
- If $\lim_{N\to\infty} s_N$ is infinite or does not exist, the series *diverges*

Unhappily, for most series, we can't compute $s_N$ explicitly. This is why we need *convergence tests*.

# A Guide to Convergence Tests

- Divergent Series Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges
- Integral test for $\sum_{n=1}^{\infty} a_n$ if $a_n = f(n)$ for a positive, continuous decreasing function $f$ (Lecture 2)
- Comparison test and limit comparison test for positive series (Lecture 3)
- Alternating series tests for 'alternating' series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ (Lecture 4)
- Ratio and Root Tests for absolute convergence of series (Lecture 6)

# Converge or Diverge?

A certain series $\sum_{n=1}^{\infty} a_n$ has the property that $\lim_{n\to\infty} a_n = 1$. Does this series converge or diverge?

A. Converge

B. Diverge

# Which Test for Which Series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+1)^2}$$

A. Divergent Series Test
B. Integral Test
C. Comparison Test
D. Limit Comparison Test
E. Alternating Series Test

$$\sum_{n=1}^{\infty} ne^{-n}$$

A. Divergent Series Test
B. Integral Test
C. Comparison Test
D. Limit Comparison Test
E. Alternating Series Test

$$\sum_{n=1}^{\infty} \frac{ne^{-1/n}}{n^3}$$

A. Divergent Series Test
B. Integral Test
C. Comparison Test
D. Limit Comparison Test
E. Alternating Series Test

# Alternating Series Test

The series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges provided:

(i) $0 \leq b_{n+1} \leq b_n$ for all $n$

(ii) $\lim_{n\to\infty} b_n = 0$

If $a_n = f(n)$ these two conditions are met if

(i) $f'(x) \leq 0$ on for $x \geq 1$

(ii) $\lim_{x\to\infty} f(x) = 0$

Remember that the AST gives a *sufficient* condition for convergence.

# Remainder Estimates

If $\sum_{n=1}^{\infty} a_n$ converges to $s$, the *remainder* $R_n$ is the difference between $s_n$ and $s$:

$$s = s_n + R_n$$

For alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$,

$$\boxed{|R_n| \leq b_n}$$

# Alternating Series Remainder Estimates

For an alternating series, $a_n = (-1)^{n-1} b_n$ and the series converges if $\{b_n\}$ is a decreasing, positive, and $\lim_{n\to\infty} b_n = 0$.

For alternating series,

$$|R_n| \leq b_n$$

How many terms of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ must be added to obtain an error of no more than 0.005?

A. $N = 200$

B. $N > 200$

C. $N < 200$

D. $N > 400$

# It's the "Tail" That Matters

A simple but important fact about series is:

If $\sum_{n=100}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

# It's the "Tail" That Matters

A simple but important fact about series is:

If $\sum_{n=100}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

And there is nothing special about 100. In fact:

If $M$ is a positive integer and $\sum_{n=M+1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

# Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

**Example 1**: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent, but not absolutely convergent. Such a series is called *conditionally convergent*.

**Example 2**: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is both convergent and absolutely convergent.

Roughly speaking a series is absolutely convergent if the convergence isn't due to cancellations between positive and negative terms.

*The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.*

Neils Hendrik Abel, 1826

The theory of infinite series was developed in the nineteenth century. Let's look at one of the paradoxes that led to the notion of *absolute convergence* that we'll study today.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \dots$$

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \dots$$

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \dots$$

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \dots$$

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

$$2S = S$$

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Dirichlet's Dilemma

In 1827, Pierre Lejeune Dirichlet discovered the following paradox while studying infinite series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \dots$$

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \dots$$

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

Thus *rearrangement* of a conditionally convergent series leads to "fallacies and paradoxes"!

Source: *Mathematics Teacher*, November 1987, Volume 80, Number 8, pp. 675–681.

# Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

**Example 1**: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent, but not absolutely convergent. Such a series is called *conditionally convergent*.

**Example 2**: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is both convergent and absolutely convergent.

Roughly speaking a series is absolutely convergent if the convergence isn't due to cancellations between positive and negative terms.

# Dirichlet's Dilemma and Riemann's Resolution

Bernhard Riemann (1826-1866) proved that a conditionally convergent series can be rearranged to sum to *any real number*!

On the other hand, it can be shown that *any rearrangment of an absolutely convergent series gives the same sum*

# Converge Conditionally, Converge Absolutely, or Diverge?

Which of the following statements is true about the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{1/2}}$ ?

A. The series converges absolutely

B. The series converges conditionally.

C. The series diverges.

# Converge Conditionally, Converge Absolutely, or Diverge?

Which of the following statements is true about the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{3/2}}$ ?

A. The series converges absolutely

B. The series converges conditionally.

C. The series diverges.

# Absolute Convergence

**Theorem** If a series $\sum a_n$ is absolutely convergent, it is also convergent.

*Proof.*

# Absolute Convergence

**Theorem** If a series $\sum a_n$ is absolutely convergent, it is also convergent.

*Proof.* First, note that

$$0 \le a_n + |a_n| \le 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$.

# Absolute Convergence

**Theorem** If a series $\sum a_n$ is absolutely convergent, it is also convergent.

*Proof.* First, note that

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$. Second, by the comparison test, $\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

# Absolute Convergence

**Theorem** If a series $\sum a_n$ is absolutely convergent, it is also convergent.

*Proof.* First, note that

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$. Second, by the comparison test, $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. Since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

we see that $\sum_{n=1}^{\infty} a_n$ converges.