Math 114 - Special Trig Integrals

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Announcement

I will be absent from UK next Wednesday, January 17 for an outpatient surgical procedure. You will have a substitute teacher in lecture.

Office hours for Wednesday, January 17 are cancelled. If you have an appointment, please see me after class to reschedule. I expect to be back on campus Friday, January 19.

Unit I: A Toolbox for Integral Calculus

Lecture 1	Integration by Parts
Lecture 2	Special Trig Integrals
Lecture 3	Trig Substitution
Lecture 4	Integrating Rational Functions, Part I
Lecture 5	Integrating Rational Functions, Part II
Lecture 6	Numerical Integration, Part I
Lecture 7	Numerical Integration, Part II
Lecture 8	Improper Integrals
Lecture 9	(Preview) Sequences
Lecture 10	(Preview) Sequences by Recursion

Review of Integration by Parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

Choose u and v so that

- u is easy to differentiate
- v is easy to integrate
- The integral $\int v(x)u'(x)\,dx$ is easier to compute

Fill in the table:

$u(x) =$ ____________	$v(x) =$ ____________
$u'(x) =$ ____________	$v'(x) =$ ____________

Uses of Integration by Parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

- Get rid of powers of x
  Examples: $\int x^2 e^x\,dx$, etc.
- Eliminate functions which have nice derivatives
  Examples: $\int x \arctan(x)\,dx$, $\int x^4 \ln x\,dx$, etc.
- Set up an equation for the unknown integral
  Example: $\int e^x \sin x\,dx$

Goals for the Day

In this lecture we'll learn techniques for computing:

- Integrals with powers of sine or cosine, like $\int \cos^3 x\, dx$, $\int \sin^3 x\, dx$, etc.
- Integrals with powers of $\tan x$ and $\cot x$, like $\int \tan^2 x\, dx$ and $\int \sec^2 x\, dx$
- Integrals with powers of sine and cosine, like $\int \sin^2 x \cos^2 x\, dx$

These techniques will involve a combination of integration formulas you already know, *trig identities*, and *algebra*.

Integrals You Already Know

$$\int \cos x \, dx = \sin x + C \qquad \int \sin x \, dx = -\cos x + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int csc^2 x \, dx = -\cot x + C$$

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The last trig identity is particularly useful because we can use it to eliminate powers of sine and cosine

New Trig Identities from Old

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Using this identity, we can find a formula for $\cos^2 x$:

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$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\cos(2x) = \cos^2 x - (1 - \cos^2 x)$$
$$\cos(2x) = 2\cos^2 x - 1$$

New Trig Identities from Old

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Using this identity, we can find a formula for $\cos^2 x$:

$$\begin{aligned}
\cos(2x) &= \cos^2 x - \sin^2 x \\
\cos(2x) &= \cos^2 x - (1 - \cos^2 x) \\
\cos(2x) &= 2\cos^2 x - 1 \\
2\cos^2 x &= 1 + \cos(2x)
\end{aligned}$$

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\cos(2x) &= \cos^2 x - (1 - \cos^2 x) \\
\cos(2x) &= 2\cos^2 x - 1 \\
2\cos^2 x &= 1 + \cos(2x) \\
\cos^2 x &= \frac{1 + \cos(2x)}{2}
\end{aligned}$$

Trig Identities

$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

Using this formula we can compute, for instance

$$\int_0^{2\pi} \cos^2 x \, dx$$

Try it!

Trig Integrals

$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Find $\int_0^{2\pi} \sin^2 x \, dx$

Switcheroo

For the rest of the hour, we'll work through the following problems together. Each one will introduce a new technique.

1. $\int \tan^2 x \, dx$ (use an identity)
2. $\int \sin^3 x \, dx$ (use u-substitution and an identity)
3. $\int \tan^3 x \, dx$ (use an identity, substitution, and what you know about $\int \tan x \, dx$
4. $\int \sin^5 x \, dx$ (use trig identity and u-substitution)
5. $\int \cos^2 x \sin^2 x \, dx$ (use trig identities including the double angle formula)

At the end we'll outline some strategies for trig integrals

Strategies for Trig Integrals

- $\int \sin^{2n+1} x \, dx$:
  Use $u = \cos x$ and $\sin^2 x = 1 - \cos^2 x$
- $\int \sin^{2n} x \, dx$ or $\int \cos^{2m} x \, dx$:
  Use double angle formulas and lots of algebra
- $\int \cos^{2n+1} x dx$:
  Use $u = \sin x$ and $\cos^2 x = 1 - \sin^2 x$
- $\int \sin^n x \cos^m x \, dx$
  Use u substitution on the 'leftover' power of sine (n odd) or cosine (m odd)