Math 114 - Trig Substitution

Peter A. Perry

University of Kentucky

January 17, 2018

Unit I: A Toolbox for Integral Calculus

Lecture 1	Integration by Parts
Lecture 2	Special Trig Integrals
Lecture 3	Trig Substitution
Lecture 4	Integrating Rational Functions, Part I
Lecture 5	Integrating Rational Functions, Part II
Lecture 6	Numerical Integration, Part I
Lecture 7	Numerical Integration, Part II
Lecture 8	Improper Integrals
Lecture 9	(Preview) Sequences
Lecture 10	(Preview) Sequences by Recursion

Strategies for Trig Integrals

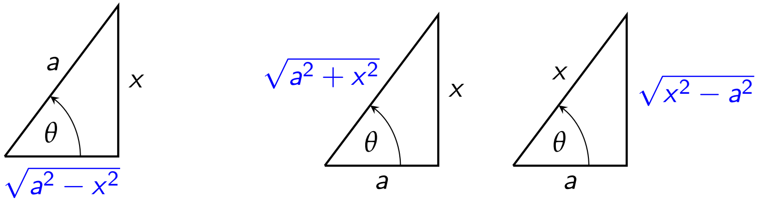
- $\int \sin^{2n+1} x\, dx$:
  Use $u = \cos x$ and $\sin^2 x = 1 - \cos^2 x$
- $\int \sin^{2n} x\, dx$ or $\int \cos^{2m} x\, dx$:
  Use double angle formulas and lots of algebra
- $\int \cos^{2n+1} x dx$:
  Use $u = \sin x$ and $\cos^2 x = 1 - \sin^2 x$
- $\int \sin^n x \cos^m x\, dx$
  Use u substitution on the 'leftover' power of sine (n odd) or cosine (m odd)

Trig Substitution

The goal of today's lecture is to learn techniques for integrating functions that involve expressions like

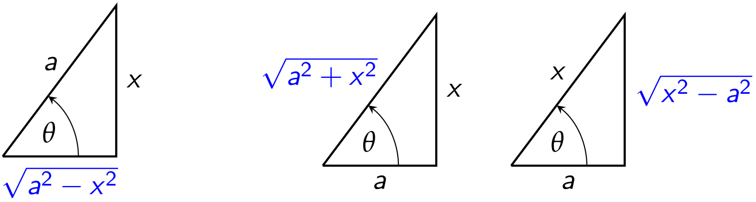
$$\sqrt{a^2 - x^2} \qquad \sqrt{a^2 + x^2} \qquad \sqrt{x^2 - a^2}$$

by changing variables to a trig function of a new variable, θ. We can think of these expressions as associated to triangles:



Outline for Trig Substitution

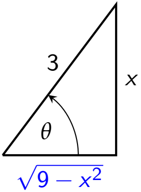
- Identify the expression $a^2 - x^2$, $a^2 + x^2$, $x^2 - a^2$ that appears in the integrand
- Draw and label the appropriate triangle
- Find x and dx in terms of θ
- Change variables to θ by substitution and compute the integral
- If an indefinite integral, use the triangle again to change variables back to x



Case Study I: An integral involving $\sqrt{9 - x^2}$

Problem: Find $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$.

Change variables to the angle θ defined by the picture:



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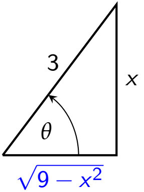

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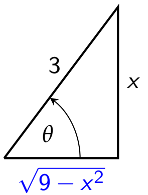

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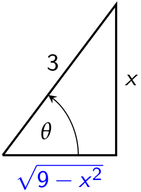

$$\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos(\theta)\,d\theta \\ \sqrt{9 - x^2} &= 3\cos\theta \end{aligned}$$

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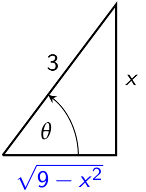

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$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

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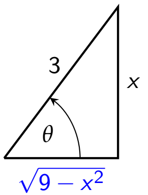

$$\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos(\theta)\, d\theta \\ \sqrt{9 - x^2} &= 3\cos\theta \end{aligned}$$

$$\begin{aligned} \int \frac{x^2}{\sqrt{9 - x^2}}\, dx &= \int \frac{9\sin^2\theta}{3\cos\theta}\, 3\cos\theta\, d\theta \\ &= \int 9\sin^2\theta\, d\theta \end{aligned}$$

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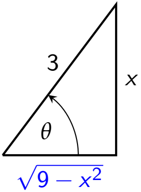

$$\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos(\theta)\,d\theta \\ \sqrt{9-x^2} &= 3\cos\theta \end{aligned}$$

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Now use what you know about computing trig integrals!

Case Study I, Continued

Problem: Find $\int \frac{x^2}{\sqrt{9 - x^2}}\,dx$

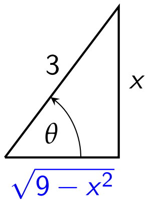
We made the substitution $x = 3\sin\theta$ to get (why?)

$$\begin{aligned}\int \frac{x^2}{\sqrt{9 - x^2}}\,dx &= \int 9\sin^2\theta\,d\theta \\ &= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C\end{aligned}$$

... but we want a formula in terms of x.

Case Study I, Continued

$$\int \frac{x^2}{\sqrt{9 - x^2}}\, dx = \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$



Now use the triangle to express θ and functions of θ in terms of x:

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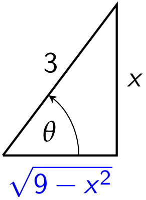

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$$x = 3\sin\theta \qquad \Longrightarrow \qquad \theta = \arcsin\left(\frac{x}{3}\right)$$

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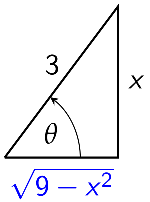

Now use the triangle to express θ and functions of θ in terms of x:

$$x = 3\sin\theta \qquad \Longrightarrow \qquad \theta = \arcsin\left(\frac{x}{3}\right)$$

$$\sin(2\theta) = 2\sin\theta \cdot \cos\theta \qquad \Longrightarrow \qquad \sin(2\theta) = 2\left(\frac{x}{3}\right) \cdot \left(\frac{\sqrt{9 - x^2}}{3}\right)$$

Case Study I, Continued

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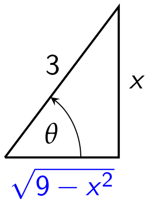

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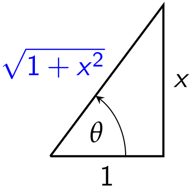
$$\frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}2\left(\frac{x}{3}\right)\cdot\left(\frac{\sqrt{9-x^2}}{3}\right)\right) + C$$

Case Study II: An Integral Involving $x^2 + 1$

Problem: Find $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx$



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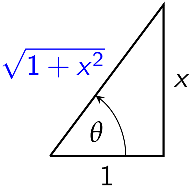

$x = \tan\theta$

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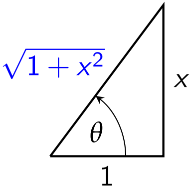

$$x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$

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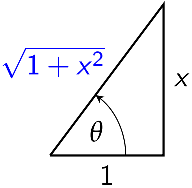

$$x = \tan\theta$$
$$dx = \sec^2\theta\,d\theta$$
$$\sqrt{1+x^2} = \sec\theta$$

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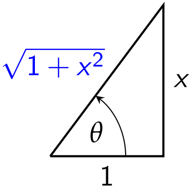

$$x = \tan\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\sqrt{1+x^2} = \sec\theta$$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta$$

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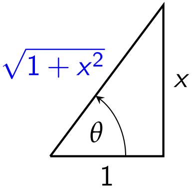
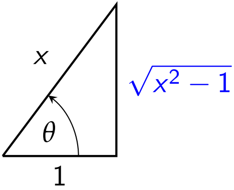


$$\begin{aligned}
x &= \tan\theta \\
dx &= \sec^2\theta\,d\theta \\
\sqrt{1+x^2} &= \sec\theta \\
\int_0^1 \frac{1}{(1+x^2)^2}\,dx &= \int_0^{\pi/4} \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta \\
&= \int_0^{\pi/4} \cos^2\theta\,d\theta = \frac{\pi}{4} = \frac{\pi}{8} + \frac{1}{4}
\end{aligned}$$

Phew!

Case Study III: An Integral Involving $\sqrt{x^2 - 1}$

Problem: Find $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x^4}\, dx$



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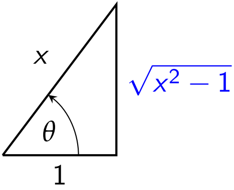

$x = \sec\theta$

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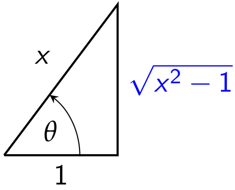

$$x = \sec\theta$$
$$dx = \sec\theta\tan\theta\,d\theta$$

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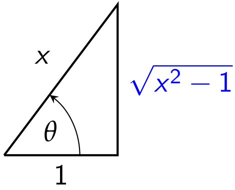

$$x = \sec\theta$$
$$dx = \sec\theta \tan\theta \, d\theta$$
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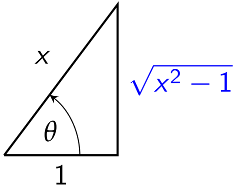

$$x = \sec\theta$$
$$dx = \sec\theta\tan\theta\,d\theta$$
$$\sqrt{x^2-1} = \tan\theta$$

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\tan\theta\;d\theta$$
$$= \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

Case Study III: Continued

Using $x = \tan\theta$ we found

$$\int \frac{\sqrt{x^2 - 1}}{x^4}\, dx = \int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

Can you simplify $\dfrac{\tan^2\theta}{\sec^3\theta}$?

Case Study III: Continued

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Can you simplify $\dfrac{\tan^2\theta}{\sec^3\theta}$?

$$\frac{tan^2\theta}{\sec^3\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta = \sin^2\theta\cos\theta$$

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Using $x = \tan\theta$ we found

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

Can you simplify $\dfrac{\tan^2\theta}{\sec^3\theta}$?

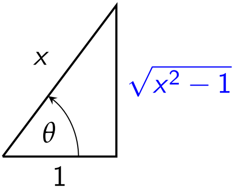
$$\frac{tan^2\theta}{\sec^3\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta = \sin^2\theta\cos\theta$$

By the substitution $u = \sin\theta$,

$$\int \sin^2\theta\,d\theta = \frac{1}{3}\sin^3\theta + C$$

Case Study III: Continued

Now change back to the x-variable using the triangle again:



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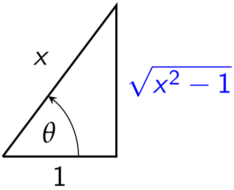

$$\sin\theta = \frac{\sqrt{x^2-1}}{x}$$

Case Study III: Continued

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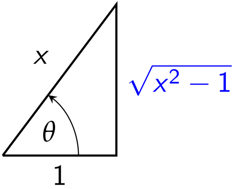

$$\sin\theta = \frac{\sqrt{x^2-1}}{x}$$

so

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \frac{1}{3}\frac{(x^2-1)^{3/2}}{x^3} + C$$

Trig Substitution: Summary of Steps

- Identify the expression $a^2 - x^2$, $a^2 + x^2$, $x^2 - a^2$ that appears in the integrand
- Draw and label the appropriate triangle
- Find x and dx in terms of θ
- Change variables to θ by substitution and compute the integral
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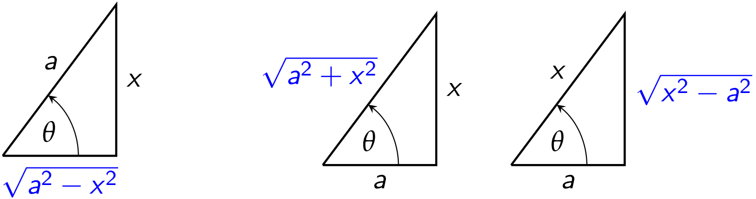

Reminders

- Homework A1 is due at 11:58 PM tonight!
- Please read section 7.4 on integration by partial fractions for Friday. We'll take two lectures on this section–the first will focus on algebra and the second on computing integrals of the partial fraction 'building blocks'