Math 114 - Integrating Rational Functions I

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Unit I: A Toolbox for Integral Calculus

REEF Reminders

- If you don't have an iClicker account, you need to get one ASAP! See the course web page for instructions; if you have any difficulty getting an account, please contact me by e-mail or see me during office hours
- Your Student ID in REEF should be your UK ID **without the "9"** or your grades in REEF will not synch with Canvas!
- Today is a "dry run" but we will begin taking attendence with REEF on *Monday*
- If you do not have a REEF compatible device (smartphone, laptop, etc.) please contact me by e-mail or see me in office hours today and we'll make another arrangement

REEF Reading Question

Which of these would be classified as an *irreducible quadratic factor*?

A. $x^2 + x - 6$

B. $x^2 + x + 1$

C. $x^2 + 2x + 1$

D. $x^2 + 10x + 25$

E. x^2

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Overview

In the next two lectures we'll develop techniques for integrating *rational functions*, that is, functions of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

We'll develop techniques to write *any* rational function as a sum of basic "building blocks" like

$$\frac{A}{(ax+b)^r}, \quad \frac{Ax+B}{(x^2+a^2)^r}$$

The process of breaking down a rational function into such building blocks is called the *partial fraction decomposition*

Example

$$\frac{5x+5}{x^2-x-6}$$

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Now we can compute

$$\int \frac{5x+5}{x^2-x-6}\,dx = \int \frac{1}{x+2}\,dx + \int \frac{4}{x-3}\,dx$$
$$= \ln|x+2| + 4\ln|x-3| + C$$

What Rational Functions can We Already Integrate?

Find the following integrals.

- $\displaystyle\int \frac{1}{3x+2}\,dx$
- $\displaystyle\int \frac{1}{x^2+9}\,dx$
- $\displaystyle\int \frac{5}{(x+4)^2}\,dx$
- $\displaystyle\int \frac{1}{x^2+x+1}\,dx$

REEF Lecture Question

Find $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

A. $\dfrac{1}{2}\arctan\left(x+1)\right)+C$

B. $\dfrac{1}{2}\arctan\left(\dfrac{x+1}{2}\right)+C$

C. $\dfrac{1}{2}\ln\left((x+1)^2+2\right)+C$

D. $\dfrac{1}{2}\arcsin\left(x+1\right)+C$

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Integrals of "Building Blocks"

You'll learn these with practice!

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \frac{1}{(ax+b)^r}\,dx = \frac{1}{a}\frac{1}{(1-r)}\frac{1}{(ax+b)^{r-1}} + C, \;\; r \neq 1$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln|x^2+a^2| + C$$

For the rest of this lecture we'll talk about:

- Dividing polynomials (review of Algebra!)
- Factoring polynomials into linear and irreducible quadratic factors (Ditto!)
- Finding the partial fraction decomposition of a rational function (New)

On Monday we'll use these tools to integrate rational functions

Dividing Polynomials

To integrate rational functions $P(x)/Q(x)$, we first need use long division for polynomials to reduce to the case where *the degree of the numerator is less than the degree of the denominator*.

Example: Suppose $f(x) = \dfrac{x^3 + 3x + 1}{x^2 + x - 6}$

The degree of the numerator is

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The degree of the numerator is 3

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Example: Suppose $f(x) = \dfrac{x^3 + 3x + 1}{x^2 + x - 6}$

The degree of the numerator is 3
The degree of the denominator is 2

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The quotient is:

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The remainder is:

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The degree of the numerator is 3
The degree of the denominator is 2

The quotient is: $x - 1$

The remainder is: $\dfrac{10x - 5}{x^2 + x - 6}$

So: $$\frac{x^3 + 3x + 1}{x^2 + x - 6} = \underbrace{(x - 1)}_{\text{polynomial}} + \underbrace{\frac{10x - 5}{x^2 + x - 6}}_{\text{rational function}}$$

Factoring Polynomials

The Fundamental Theorem of Algebra says that a polynomial of degree n with real coefficients has n zeros. These zeros can be:

- Real zeros, corresponding to *linear factors* $x - a$
- Complex zeros (in complex conjugate pairs) corresponding to *irreducible quadratic factors* $ax^2 + bx + c$ where $b^2 - 4ac < 0$

We'll now look at some example of factoring polynomials

Factoring Polynomials

Factor the following polynomials into linear factors $(x-a)$ and irreducible quadratic factors ax^2+bx+c.

1. $x^2-2x-15$
2. x^2+x+1
3. x^3-2x^2
4. x^4+15x^2

To find the partial fraction decomposition of a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

1. If the degree of P is bigger than the degree of Q, do polynomial long division to write P/Q as a polynomial plus a remainder
2. Factor the denominator $Q(x)$ into linear factors and irreducible quadratic factors
3. Write $P(x)/Q(x)$ as a sum of 'building blocks' with coefficients to be determined using the table below

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Repeated Linear $(ax+b)^r$	$\frac{A_1}{ax-b} + \cdots + \frac{A_r}{(ax+b)^r}$
Irreducible Quadratic (ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
Repeated Irreducible Quadratic $(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$