Math 114 - Integrating Rational Functions II

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January 22, 2018

Unit I: A Toolbox for Integral Calculus

Overview

This is the second of two lectures where we develop techniques for integrating *rational functions*, that is, functions of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

We'll develop techniques to write *any* rational function as a sum of basic "building blocks" like

$$\frac{A}{(ax+b)^r}, \quad \frac{Ax+B}{(x^2+a^2)^r}$$

The process of breaking down a rational function into such building blocks is called the *partial fraction decomposition*

To find the partial fraction decomposition of a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

1. If the degree of P is bigger than the degree of Q, do polynomial long division to write P/Q as a polynomial plus a remainder
2. Factor the denominator $Q(x)$ into linear factors and irreducible quadratic factors
3. Write $P(x)/Q(x)$ as a sum of 'building blocks' with coefficients to be determined using the table below

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Repeated Linear $(ax+b)^r$	$\frac{A_1}{ax-b} + \cdots + \frac{A_r}{(ax+b)^r}$
Irreducible Quadratic (ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
Repeated Irreducible Quadratic $(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Use the table to find the form of the PDF for each of the following rational functions Dont't try to solve for the coefficients... yet

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Repeated Linear $(ax+b)^r$	$\frac{A_1}{ax+b}+\cdots+\frac{A_r}{(ax+b)^r}$
Irreducible Quadratic (ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
Repeated Irreducible Quadratic $(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$

1. $\dfrac{5x+10}{x^2-2x-15}$
2. $\dfrac{3x+2}{(x^2+x+1)^2}$
3. $\dfrac{5x^2-4x-4}{x^3-2x^2}$
4. $\dfrac{2x+1}{x^2+2x+1}$

How to Find the Coefficients, Part I

Factor in $Q(x)$	Term in PFD
Linear $(ax+b)$	$\dfrac{A}{ax+b}$

According to the table,

$$\frac{5x+10}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5}$$
$$= \frac{A(x-5)+B(x+3)}{(x+3)(x-5)}$$

How do we find the numbers A and B? Focus on the *numerator* on left and right:

$$\frac{5x+10}{(x+3)(x-5)} = \frac{A(x-5)+B(x+3)}{(x+3)(x-5)}$$

Set $x=5$ and $x=-3$ on both sides to get two linear equations for A and B

How To Find the Coefficients, Part I

How do we find the numbers A and B? Focus on the *numerator* on left and right:

$$\frac{5x+10}{(x+3)(x-5)} = \frac{A(x-5)+B(x+3)}{(x+3)(x-5)}$$

Set $x=5$ and $x=-3$ on both sides to get two linear equations for A and B

$$\begin{aligned} 35 &= 8B && \text{(from } x=5\text{)} \\ -5 &= -8A && \text{(from } x=-3\text{)} \end{aligned}$$

So

$$\frac{5x+1}{(x+3)(x-5)} = \frac{5}{8}\frac{1}{x+3} + \frac{35}{8}\frac{1}{x-5}$$

How To Find the Coefficients, Part II

Factor in $Q(x)$	**Term in PFD**
Repeated Irreducible Quadratic $(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$

According to the table

$$\begin{aligned}\frac{3x+2}{(x^2+x+1)^2} &= \frac{A_1x+B_1}{(x^2+x+1)}+\frac{A_2x+B_2}{(x^2+x+1)^2}\\ &= \frac{(A_1x+B_1)(x^2+x+1)+(A_2x+B_2)}{(x^2+x+1)^2}\end{aligned}$$

In this case, a better strategy is to collect powers of x in the numerator:

$$\frac{3x+2}{(x^2+x+1)^2}=\frac{A_1x^3+(A_1+B_1)x^2+(A_1+B_1+A_2)x+(A_1+B_1+B_2)}{(x^2+x+1)^2}$$

and notice that $A_1=0$ (degree 3), $A_1+B_1=0$ (degree 2), so $A_1=B_1=0$!

What are A_2 and B_2?

The Story So Far

To write a rational function $f(x) = P(x)/Q(x)$ in terms of the 'building blocks':

- If needed, divide $P(x)$ by $Q(x)$ to express $f(x)$ as a polynomial plus a remainder where the degree of the numerator is *less* than the degree of the denominator. From now on, we'll assume this has already been done.
- Factor the denominator $Q(x)$ into linear factors and irreducible quadratic factors
- Use the table to write down the partial fraction decomposition of f with coefficients to be determined
- Put the terms in the PFD over a common denominator
- Solve for the unknown coefficients either by substituting special values of x or by collecting powers

Now You Try It

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Repeated Linear $(ax+b)^r$	$\frac{A_1}{ax+b}+\cdots+\frac{A_r}{(ax+b)^r}$

Find the partial fraction decomposition of

$$f(x)=\frac{5x^2-4x-4}{x^3+2x^2+x}$$

and use it to compute

$$\int \frac{5x^2-4x-4}{x^3+2x^2+x}\,dx$$

Now You Try It

Factor in $Q(x)$	**Term in PFD**
Linear $(ax+b)$	$\frac{A}{ax+b}$
Irreducible Quadratic (ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$

Find the partial fraction decomposition of

$$f(x) = \frac{2x+4}{x^3+x^2+x}$$

and use it to compute

$$\int \frac{2x+4}{x^3+x^2+x}\,dx$$

Now You Try it on WebWork

You should be finishing up Webwork A2 and you should begin Webwork A3. Stay ahead of the game and try some problems from Webwork A3 tonight!

Upcoming Deadlines:

- Read section 7.7 for Wednesday January 25
- Webwork A2 due by 11:50 Wednesday January 24
- Study for Quiz 2 for Thursday
- Webwork A3 due by 11:58 Friday January 26
- On the horizon: Test I, February 6, 5:00-7:00 PM