Math 114 - Numerical Integration I

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January 24, 2018

Unit I: A Toolbox for Integral Calculus

This is the first of two lectures on computing *definite* integrals *numerically*. We'll turn the idea of approximating an integral by Riemann sums into a way of computing the definite integral

$$\int_a^b f(x)\,dx$$

In all of these methods, we will:

- Divide the interval $[a, b]$ into N equal subintervals
- Sample one value of $f(x)$ in each interval
- Multiply each sampled value by an appropriate weight
- Add up all of the weights $\times$ samples to approximate the integral

We'll approximate $\int_a^b f(x)\,dx$ by a finite sum

$$\sum_{i=1}^{N} w_i f(x_i)$$

where w_i are *weights* and x_i are *sample points* in each of N intervals

We'll introduce two old and three new numerical methods:

- *Left* and *Right* Riemann sums
- The *midpoint rule*, which approximates the function f on each interval by a constant function
- The *trapezoid rule*, which approximates the function f on each interval by a linear function
- *Simpson's rule*, which approximates the function on each *two* intervals by a quadratic function

We'll also learn how to estimate the likely accuracy of each numerical method through *error estimates* which are like a 'guarantee' for each numerical method

Finally, we'll look at circumstances under which each method fails miserably to give a reasonable answer!

Left and Right Riemann Sums

To approximate $\int_a^b f(x)\,dx$, let

$$\Delta x = \frac{b-a}{n} \quad x_i = a + i\Delta x$$

The *Left Riemann Sum* is

$$L_n = \sum_{i=1}^{N} \underbrace{f(x_{i-1})}_{\text{left endpoint}} \Delta x_i$$

The *Right Riemann Sum* is

$$R_n = \sum_{i=1}^{N} \underbrace{f(x_i)}_{\text{right endpoint}} \Delta x_i$$

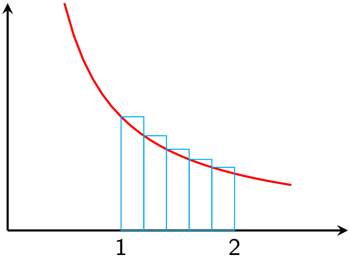
Left Riemann Sums

Suppose we want to approximate $\ln 2 \simeq 0.6931$ by computing

$$\int_1^2 \frac{1}{x}\, dx.$$

One way to do this is via a *left Riemann sum* say for 5 subintervals of $[1, 2]$ so $\Delta x = 0.2$

x	$1/x$
1	1.0000
1.2	0.8333
1.4	0.7143
1.6	0.6250
1.8	0.5556



We get an overestimate

$$\begin{aligned} L_5 &= 0.2 \times (1.0000 + 0.8333 + 0.7143 + 0.6250 + 0.5556) \\ &= 0.7456 \end{aligned}$$

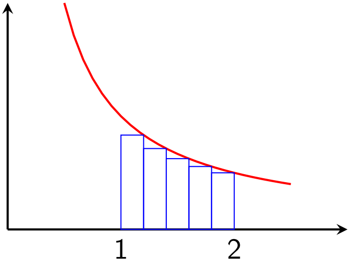
Right Riemann Sums

We could also try to estimate

$$\int_1^2 \frac{1}{x}\,dx \simeq 0.6931$$

using a *right Riemann sum* for the same five subintervals

x	$1/x$
1.2	0.8333
1.4	0.7143
1.6	0.6250
1.8	0.5556
2.0	0.5000

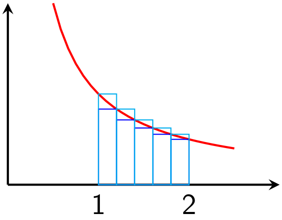


We get an underestimate

$$\begin{aligned} R_5 &= 0.2 \times (0.8333 + 0.7143 + 0.6250 + 0.5556 + 0.5000) \\ &= 0.6456 \end{aligned}$$

This Suggests a Compromise...

For a decreasing function, the left Riemann sum always makes an *overestimate* of the integral, and the right Riemann sums always makes an *underestimate*:



- Sampling on the left gives an overestimate
- Sampling on the right gives an underestimate
- So why not sample at the midpoint instead?

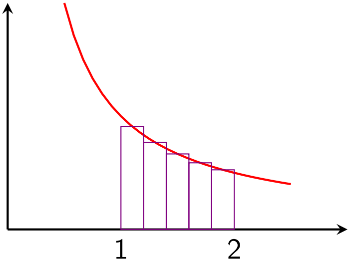
The Midpoint Method

Let's try to estimate

$$\int_1^2 \frac{1}{x}\, dx \simeq 0.6931$$

by sampling at midpoints of each interval

x	$1/x$
1.1	0.9091
1.3	0.7692
1.5	0.6667
1.7	0.5882
1.9	0.5263



This time we get a much more accurate estimate

$$M_5 = 0.2 \times (0.9091 + 0.7692 + 0.6667 + 0.5882 + 0.5263)$$
$$= 0.6919$$

Left Endpoint, Right Endpoint, and Midpoint Methods

We've seen three methods for computing

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = (b-a)/n$ and x_i^* is the sampling point for the ith interval

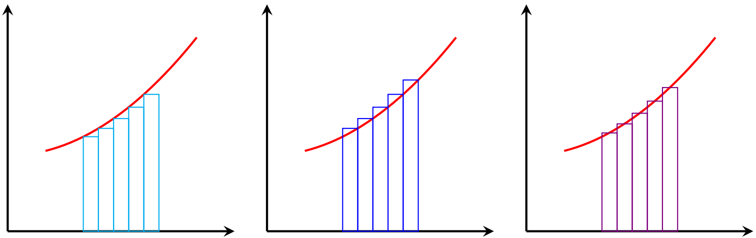
- The left endpoint sum L_n, where $x_i^* = x_{i-1}$
- The right endpoint sum R_n, where $x_i^* = x_i$
- The midpoint sum M_n, where $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

Food for Thought

Which of the following is the correct order?

A. $L_n \leq R_n \leq M_n$ B. $R_n \leq M_n \leq L_n$ C. $L_n \leq M_n \leq R_n$

D. $M_n \leq R_n \leq L_n$ E. $M_n \leq L_n \leq R_n$



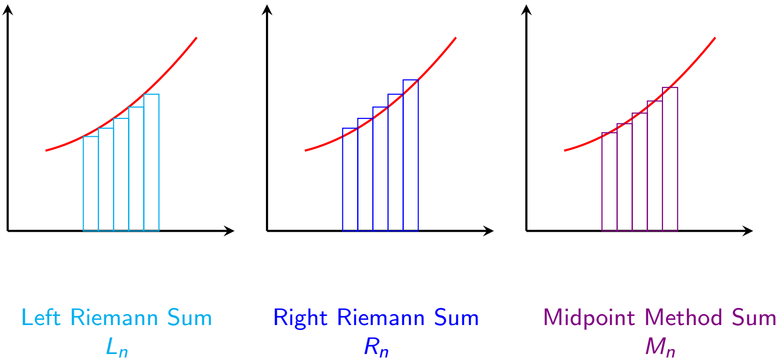
Left Riemann Sum L_n | Right Riemann Sum R_n | Midpoint Method Sum M_n

Food for Thought

Which of the following is the correct order?

A. $L_n \leq R_n \leq M_n$ B. $R_n \leq M_n \leq L_n$ C. $L_n \leq M_n \leq R_n$

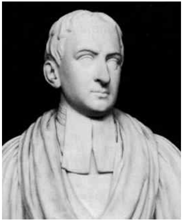
D. $M_n \leq R_n \leq L_n$ E. $M_n \leq L_n \leq R_n$



Newton and Cotes Show You A Better Way

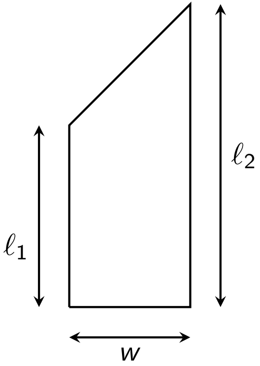
Sir Isaac Newton (1643-1727), Lucasian Professor, Cambridge University

The next two numerical methods we'll discuss (the trapezoid method and Simpson's method) were developed by Isaac Newton and his colleague Roger Cotes, although they were also known to others. For this reason they're sometimes called *Newton-Cotes Formulas*.



Roger Cotes (1682-1716), Plumian Professor, Cambridge University

From Boxes to Trapezoids

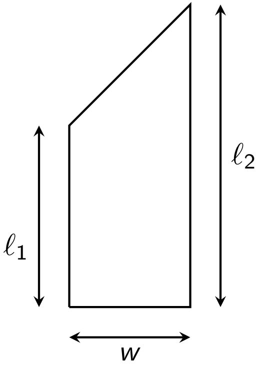


Which of the following is the correct formula for the area of the trapezoid shown at left?

A. $w(\ell_1 + \ell_2)$
B. $\pi w \ell_1 \ell_2$
C. $\frac{w}{2}(\ell_1 + \ell_2)$
D. $\frac{1}{2} w \ell_1 \ell_2$
E. None of the above

From Boxes to Trapezoids

A yet better way to approximate an integral uses the formula for the area of a trapezoid:



$$A = h \cdot \frac{1}{2}\left(\ell_1 + \ell_2\right)$$

From Boxes to Trapezoids

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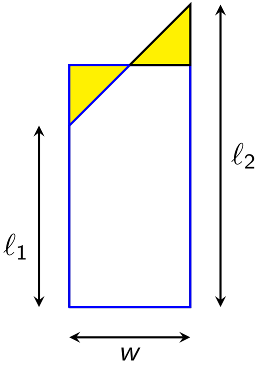
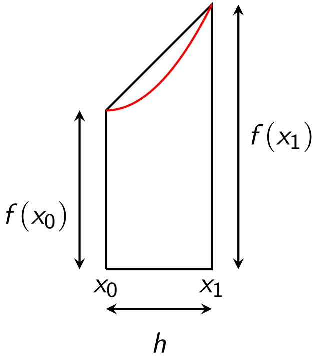


$$A = h \cdot \frac{1}{2}(\ell_1 + \ell_2)$$

You can see this by noting that the trapezoid has the same area as a rectangle with width w and height $(\ell_1 + \ell_2)/2$

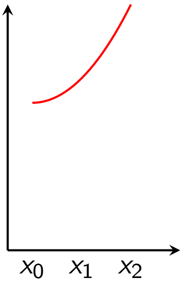
The Trapezoid Method for One Interval



$$\int_{x_0}^{x_1} f(x)\,dx \simeq \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

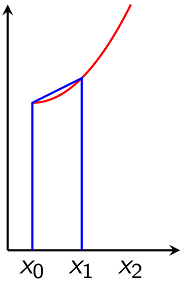
Notice that we've renamed Δx as h.

The Trapezoid Rule for Two Intervals



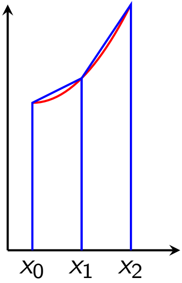
$$\int_{x_0}^{x_2} f(x)\, dx$$

The Trapezoid Rule for Two Intervals



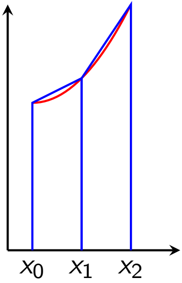
$$\int_{x_0}^{x_2} f(x)\,dx \simeq \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

The Trapezoid Rule for Two Intervals



$$\int_{x_0}^{x_2} f(x)\,dx \simeq \frac{h}{2}\left(f(x_0) + f(x_1)\right) + \frac{h}{2}\left(f(x_1) + f(x_2)\right)$$

The Trapezoid Rule for Two Intervals

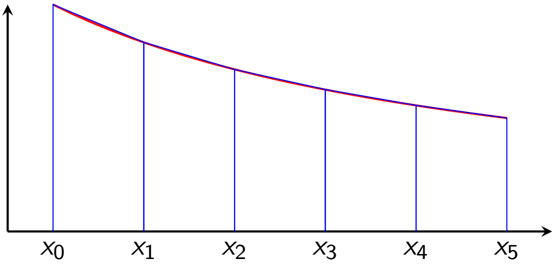


$$
\begin{aligned}
&\int_{x_0}^{x_2} f(x)\,dx \\
&\qquad \simeq \frac{h}{2}\left(f(x_0) + f(x_1)\right) \\
&\qquad\quad + \frac{h}{2}\left(f(x_1) + f(x_2)\right) \\
&\qquad = \frac{h}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right)
\end{aligned}
$$

TheTrapezoid Rule for 5 Intervals

$$\int_{x_0}^{x_5} f(x)\,dx =$$

$$\frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$



The Trapezoid Rule for $\int_1^2 \frac{1}{x}\,dx$

$$T_5 = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

x	$1/x$	Weight	$h/2$	Product
1.0	1.0000	1	0.1	0.1000
1.2	0.8333	2	0.1	0.1667
1.4	0.7143	2	0.1	0.1429
1.6	0.6250	2	0.1	0.1250
1.8	0.5556	2	0.1	0.1111
2.0	0.5000	1	0.1	0.0500
Sum				0.5956

Sum	0.6956
True Value	0.6931
Error	0.0025

All The Rules So Far

To compute $\int_a^b f(x)\,dx$, set $x_0 = a$, $h = (b-a)/n$, $x_i = x_0 + ih$

$$L_n = h\left[f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})\right]$$

$$R_n = h\left[\qquad f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)\right]$$

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

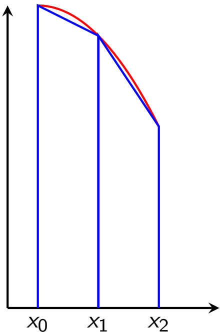
Notice that the trapezoid method is the compromise

$$T_n = \tfrac{1}{2}\left(L_n + R_n\right)$$

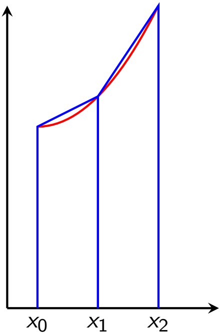
Underestimating and Overestimating

$$I = \int_a^b f(x)\,dx$$

If the graph of f is *concave down*, the trapezoid rule T_n will always *underestimate* I



If the graph of f is *concave up*, the trapezoid rule T_n will always *overestimate* I



What's the Right Order?

Suppose we wish to approximate

$$I = \int_a^b f(x)\,dx$$

for the function shown at right. Which of the following statements is true about L_n, R_n, T_n, and I?

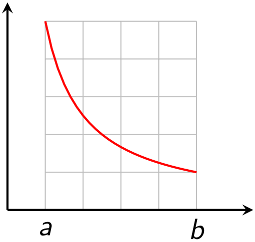
A. $L_n \leq I \leq R_n \leq T_n$

B. $L_n \leq I \leq T_n \leq R_n$

C. $R_n \leq I \leq T_n \leq L_n$

D. $R_n \leq I \leq L_n \leq T_n$

E. $R_n \leq L_n \leq I \leq T_n$



What's the Right Order?

Suppose we wish to approximate

$$I = \int_a^b f(x)\,dx$$

for the function shown at right. Which of the following statements is true about L_n, R_n, T_n, and I?

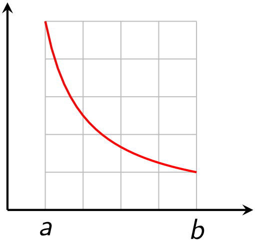
A. $L_n \leq I \leq R_n \leq T_n$

B. $L_n \leq I \leq T_n \leq R_n$

C. $R_n \leq I \leq T_n \leq L_n$

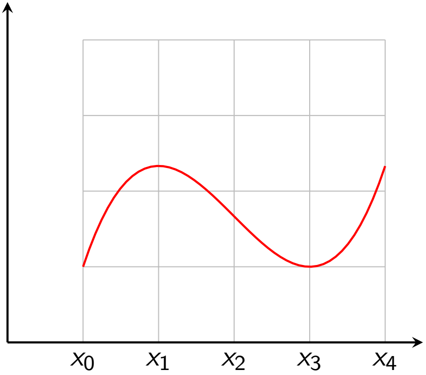
D. $R_n \leq I \leq L_n \leq T_n$

E. $R_n \leq L_n \leq I \leq T_n$



From Trapezoids to Parabolas

Simpson's Rule approximates the function to be integrated over a *pair of intervals* by a *quadratic* function



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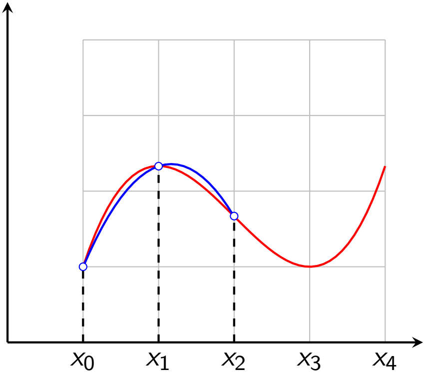

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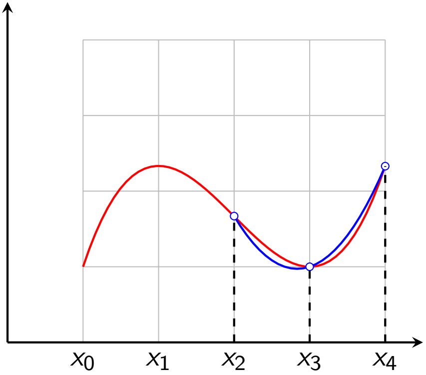


How to Integrate a Quadratic Function

$$\int_{-h}^{h} \left(ax^2 + bx + c\right)\, dx = \frac{2a}{3}h^3 + 2ch$$

How to Fit Any Function to a Quadratic Function

$$\begin{aligned} f(-h) &= ah^2 - bh + c \\ f(0) &= c \\ f(h) &= ah^2 + bh + c \end{aligned}$$

These three equations give

$$\begin{aligned} a &= \frac{1}{2h^2}\left(f(-h) + f(h) - 2f(0)\right) \\ b &= \frac{1}{2h}\left(f(h) - f(-h)\right) \\ c &= f(0) \end{aligned}$$

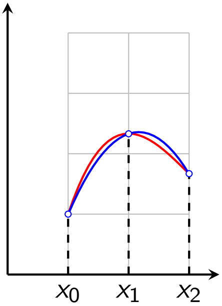
Simpson's Rule for Two Intervals

Put together the formula

$$\int_{-h}^{h} (ax^2 + bx + c)\, dx = \frac{2a}{3} h^3 + 2ch$$

and fit a, b, and c to the values of f, and you get

$$\int_{x_0}^{x_2} f(x)\, dx \simeq \frac{h}{3} \left[f(x_0) + 4f(x_1) + f(x_2) \right]$$



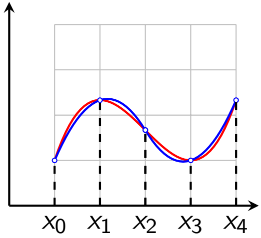
Simpson's Rule for Four Intervals

For *four* intervals we can apply the rule

$$\int_{x_0}^{x_2} f(x)\,dx \simeq \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

twice to get

$$\int_{x_0}^{x^4} f(x)\,dx \simeq \frac{h}{3}\left[(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$



Simpson's Rule and $\int_1^2 (1/x)\, dx$

$$S_4 = \frac{h}{3}\left[(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

x	$1/x$	Weight	$h/3$	Product
1.0	1.0000	1	0.0833	0.0833
1.25	0.8333	2	0.0833	0.2667
1.5	0.7143	4	0.0833	0.1111
1.75	0.6250	2	0.0833	0.1905
2.0	0.5000	1	0.0833	0.0417
Sum				0.6933

Sum	0.6933
True Value	0.6931
Error	0.0001

Moral: Simpson's Rule Rules!

Now You Try it on WebWork

You should be finishing up Webwork A2 and you should begin Webwork A3. Stay ahead of the game and try some problems from Webwork A3 tonight!

Upcoming Deadlines:

- Re-Read section 7.7 for Friday January 26
- Webwork A2 due by 11:50 tonight, Wednesday January 24
- Study for Quiz 2 for Thursday covering Trig Integrals and Trig Substitution
- Webwork A3 due by 11:58 Friday January 26
- On the horizon: Test I, February 6, 5:00-7:00 PM