Math 114 - Numerical Integration II

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Unit I: A Toolbox for Integral Calculus

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Lecture 3	Trig Substitution
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Lecture 10	(Preview) Sequences by Recursion

This is the second of two lectures on computing *definite* integrals *numerically*. We've now seen the following methods of computing $I = \int_a^b f(x)\,dx$ numerically:

- The left endpoint method L_n (left Riemann sum)
- The right endpoint method R_n (right Riemann sum)
- The midpoint method M_n
- The trapezoid method T_n
- Simpson's method S_n

In each of these methods we divide up $[a, b]$ into n intervals of size h and sample f at n points. We then weight these values and sum up values $\times$ weights to approximate I.

We'll concentrate today on:

- The midpoint method,

$$M_n = \sum_{i=1}^{n} hf(x_i^*), \quad x_i^* = \frac{1}{2}(x_{i-1} + x_i)$$

- The trapezoid method

$$T_n = \frac{h}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))$$

- Simpson's rule (which works only when n is *even*)

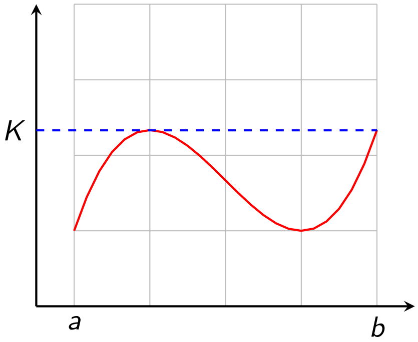
$$S_n = \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \\ + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

We will see what happens, for each method, as we increase n, the number of intervals, and see how to insure that a numerical computation is accurate to a given number of decimal places. We can see this by measuring the *absolute error* $E_* = |I - I_*|$ where I is the true value and I_* is the computed value.

To do this we'll state *error estimates* for each of the three methods M_n, T_n, and S_n which are a "guarantee" that, for a given function f, interval $[a, b]$, and n, the difference between the computed and exact values is *no more than* a certain number.

Pause for New Word

A number K is an *upper bound* for a function $g(x)$ on $[a, b]$ if $g(x) \leq K$ for every $x \in [a, b]$.



Find an upper bound for the function xe^x on the interval $[0, 2]$.

A. 0
B. e
C. $2e$
D. $2e^2$
E. $3e$

An Experiment with the Midpoint Rule

Here's how the midpoint rule applied to $\int_0^1 (1/x)\,dx$ improves as we increase n, the number of subintervals

	Exact	**Computed**	**Error**	**Ratio**
M_5	0.693147	0.691908	0.001239	
M_{10}	0.693147	0.692835	0.000312	0.2516
M_{20}	0.693147	0.693069	0.000078	0.2504
M_{40}	0.693147	0.693128	0.000020	0.2501

What happens as n increases? What do you notice about the ratio of successive errors?

Midpoint Rule Warranty

The data indicate that the absolute error decreases like n^{-2} as n, the number of intervals, increases. Here is the official warranty.

Midpoint Method Warranty If the midpoint method is used to approximate $I = \int_a^b f(x)\,dx$ with n intervals of size $h = (b-a)/n$, the absolute error

$$E_M = |I - M_n|$$

will never be more than

$$\frac{K(b-a)^3}{24n^2}$$

where K is an upper bound for $|f''(x)|$ on $[a, b]$.

How to Use the Warranty

The midpoint method warranty guarantees that the difference between the *true* value of $\int_a^b f(x)\,dx$ and the *computed* value M_n will never be more than

$$\frac{K(b-a)^3}{24n^2}$$

We can use this to choose n large enough to get as many decimal places of accuracy as we wish. The steps are:

- Compute $f''(x)$
- Find an upper bound for $|f''(x)|$ on $[a, b]$
- Find n so that the error will be no greater than the desired accuracy

How to Use the Warranty $E_M \leq K(b-a)^3/(24n^2)$

Suppose we want to compute $\int_0^1 e^{-x^2}\,dx$ to an accuracy of 0.00001 using the midpoint method.

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- Since $f(x) = e^{-x^2}$, $f''(x) = e^{-x^2}(4x^2 - 2)$. On $[0,1]$, the maximum value of $|f''(x)|$ is 2 (you can check this by calculus; the derivative of $f''(x)$ is $-4e^{-x^2}x(2x^2-3)$ so critical points are $x = 0$, $x = \pm\sqrt{3/2}$)

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- With $K=2$, $a=0$, $b=1$, the warranty guarantees that the error will be *no more than* $2\cdot(1-0)^3/(24n^2) = 1/(12n^2)$

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- With $K=2$, $a=0$, $b=1$, the warranty guarantees that the error will be *no more than* $2\cdot(1-0)^3/(24n^2) = 1/(12n^2)$
- So we want to choose n large enough that

$$\frac{1}{12n^2} < 0.00001$$

How to Use the Warranty $E_M \leq K(b-a)^3/(24n^2)$

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so

$$12n^2 > 100,000$$

or

$$n > 91.29$$

so $n = 92$ or greater will do.

How many intervals n must we use to compute $\int_0^1 \sin(40x)\,dx$ using the midpoint method to an accuracy of 0.0001? Use the error bound

$$E_M \leq \frac{K(b-a)^3}{24n^2}$$

where K is an upper bound for $f''(x)$ on $[0,1]$

A. 204

B. 205

C. 408

D. 817

E. 1634

What the Warranty Means

$$E_M \leq \frac{K(b-a)^3}{24n^2}, \quad |f''(x)| \leq K \text{ on } [a,b]$$

- The larger the number of intervals, n, the lower the error
- The larger the interval $b - a$, the larger the error
- The more oscillatory the function is, the larger the error

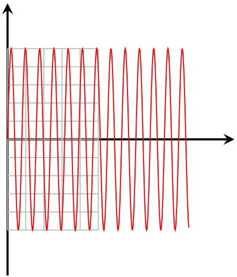
What the Warranty Means, Part II

$$E_M \leq \frac{K(b-a)^3}{24n^2}, \quad |f''(x)| \leq K \text{ on } [a,b]$$

$|f''(x)|$ (and so K) is a measure of how highly oscillating the function f is. The function

$$f(x) = \sin(40x)$$

is a rapidly oscillating function. For this reason, we need many small intervals to compute the integral correctly.



An Experiment with Simpson's Rule

We'll try Simpson's rule with 2, 4, and 8 intervals on $\int_1^2 (1/x)\, dx$.

n	S_n	**Exact**	**Error**
2	0.69444444	0.69314718	0.0012972639
4	0.69325397	0.69314718	0.0001067877
8	0.69315453	0.69314718	0.0000073501

Notice that even 2 intervals gives three decimal place accuracy!

Simpson's Rule Warranty

If $I = \int_a^b f(x)\,dx$, the error $E_S = |I - S_n|$ is at most

$$\frac{K(b-a)^5}{180n^4}, \quad |f^{(iv)}(x)| \leq K \text{ on } [a,b]$$

- A larger interval leads to larger errors
- A larger fourth derivative (i.e., more oscillation) leads to larger errors
- A larger number of intervals leads to smaller errors
- Doubling the number of intervals should decrease the interval by a factor of 16!

Puzzler

$$E_S \leq \frac{K(b-a)^5}{180n^4}, \quad |f^{(iv)}(x)| \leq K \text{ on } [a,b]$$

What is the *maximum possible error* in using Simpson's rule to compute $\int_0^2 (x^3 + 3x^2 + x + 1)\,dx$?

A. 1/180
B. 0
C. 64/180
D. 1/90
E. None of the above

All of the Warranties on One Page

Midpoint Rule	$E_M \leq \dfrac{K(b-a)^3}{24n^2}$	K bounds $\lvert f''(x)\rvert$ on $[a,b]$
Trapezoid Rule	$E_T \leq \dfrac{K(b-a)^3}{12n^2}$	K bounds $\lvert f''(x)\rvert$ on $[a,b]$
Simpson's Rule	$E_S \leq \dfrac{K(b-a)^5}{180n^4}$	K bounds $\lvert f^{(4)}(x)\rvert$ on $[a,b]$

Suppose we approximate $\int_0^2 \cos(10x)\,dx$ using M_8, T_8, and S_8. What is the maximum error for each method? Note that

$$f(x) = \cos(10x), \quad f''(x) = -100\cos(10x), \quad f^{(iv)}(x) = 10000\cos(10x)$$