Math 114 - Improper Integrals

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Unit I: A Toolbox for Integral Calculus

Overview

In this lecture we return to computing integrals by *analytic* rather than *numeric* techniques, but applied to two new types of integrals:

- Improper integrals of type I, where one or more of the limits of integration is infinite

  Example: $\int_0^\infty e^{-x}\,dx$

- Improper integrals of type II, where the integrand is not continuous at one of the endpoints or within the interval of integration

  Example: $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

Overview

Remember that, so far, we've only defined integrals $\int_a^b f(x)\,dx$ when a and b are both *finite* and $f(x)$ is continuous. So, before we *compute* improper integrals, we have to *define* what we mean!

Preview: We'll define improper integrals as *limits* of 'proper' integrals

Infinite Limits

Find $\lim_{t\to\infty}\left(3-\frac{2}{t}\right)$, $\lim_{t\to\infty}\left(1-e^{-t}\right)$, and $\lim_{x\to 1^-}\sqrt{1-x}$

A. 3, 1, 0
B. 1, 3, 0
C. 0, 1, 3
D. 2, 0, 0
E. 0, 3, 1

Limits Review

Find $\lim_{t\to\infty}\left(3-\frac{2}{t}\right)$, $\lim_{t\to\infty}\left(1-e^{-t}\right)$, and $\lim_{x\to 1^-}\sqrt{1-x}$

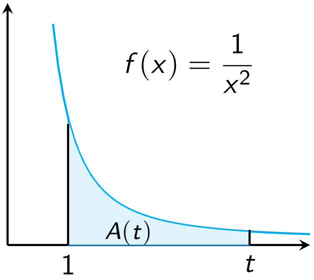
A. 3, 1, 0
B. 1, 3, 0
C. 0, 1, 3
D. 2, 0, 0
E. 0, 3, 1

Type I: Infinite Intervals

The area under the graph of $f(x) = 1/x^2$ from $x = 1$ to $x = t$ is:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = 1 - \frac{1}{t}$$

What happens as we increase t?



t	2	3	4	5	6	7	8	100
$A(t)$	$1/2$	$2/3$	$3/4$	$4/5$	$5/6$	$6/7$	$7/8$	$99/100$

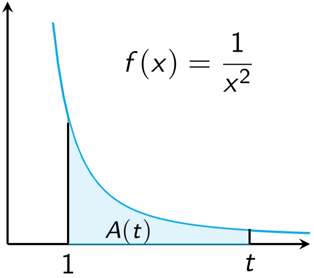
Type I: Infinite Intervals

The area under the graph of $f(x) = 1/x^2$ from $x = 1$ to $x = t$ is:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = 1 - \frac{1}{t}$$

This expression has a *finite limit* as $t \to \infty$ so we define

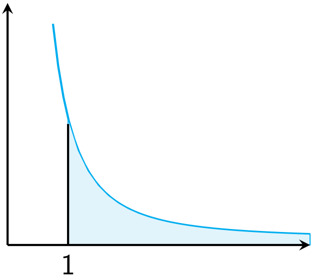
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty} \left(1 - \frac{1}{t}\right)$$
$$= 1$$



Type I: Infinite Intervals

So, the remarkable fact is that the *total* area under the curve from $x = 1$ out to $+\infty$ is actually *finite*!

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1$$



Type I: Infinite Intervals

In the same way we can define integrals over any semi-infinite or infinite interval:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$
$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

If the limit exists, we say that the improper integral *converges*. If the limit does not exist, we say that the improper integral *diverges*.

Now You Try It

Determine whether the improper integral $\int_0^\infty e^{-x}\,dx$ converges or diverges. If it converges, find its value. Remember that

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

if the limit exists.

A. Diverges
B. Converges, value e
C. Converges, value 1
D. Converges, value -1
E. Converges, value 0

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Now You Try It

Determine whether the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges. If it converges, find its value. It may help to remember that

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

A. Diverges

B. Converges, value $\pi/2$

C. Converges, value π

D. Converges, value 2π

E. Converges, value 1

Now You Try It

Determine whether the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ converges. If it converges, find its value. It may help to remember that

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Culture Break: Probability Theory

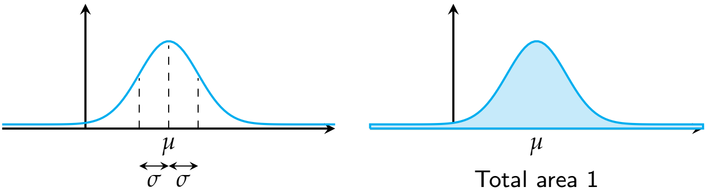
Many measured quantities x obey the *normal distribution* (the "bell-shaped curve") with *mean* or average value μ and *standard deviation* σ. The probability that x lies between a and b is

$$P(a < x < b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

For this to make sense it had better be true that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = 1$$

since the *total* probability that the measurement x falls *somewhere* on the real line had better be 1!



A second type of improper integral is one where the integrand is discontinuous.

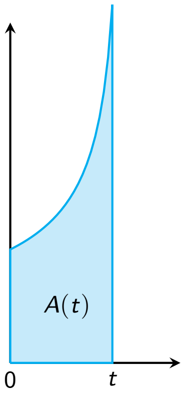
To determine

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$

we'll consider the area function

$$A(t) = \int_0^t \frac{1}{\sqrt{1-x}}\,dx$$

and study its behavior as $t \to 1^-$.



Use u-substitution to find $\int_0^t \frac{1}{\sqrt{1-x}}\,dx$

A. $2\left(\sqrt{1-t}-1\right)$
B. $2\left(1-\sqrt{1-t}\right)$
C. $\frac{1}{2}\left(\sqrt{1-t}-1\right)$
D. $\frac{1}{2}\left(1-\sqrt{1-t}\right)$
E. $2\sqrt{1-t}+C$

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A second type of improper integral is one where the integrand is discontinuous.

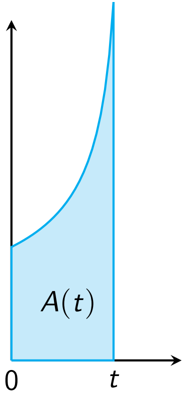
To determine

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$

we'll consider the area function

$$A(t) = \int_0^t \frac{1}{\sqrt{1-x}}\,dx = 2\left(1-\sqrt{1-t}\right)$$

If $\lim_{t\to 1^-} A(t)$ exists, then this type II integral converges. Does it, and if so to what?



Type II: Discontinuous Integrands

- If f is discontinuous at $x = a$ but continuous in $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

- If f is discontinuous at $x = b$ but continuous in $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

- If f is continuous on $[a, c)$ and $(c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

  if the two improper integrals converge.

As always, an improper integral *converges* if the limit that defines it exists, and *diverges* otherwise.

Type II: Discontinuous Integrands

Determine whether the Type II improper integral

$$\int_0^1 \log(x)\,dx$$

converges or diverges. It should help to know that

$$\int \log(x)\,dx = x\log x - x + C$$

and to know that $\lim_{x\to 0^+} x\log x = 0$.

A. Diverges

B. Converges, value 0

C. Converges, value -1

D. Converges, value -2

E. None of the above

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C. Converges, value -1

D. Converges, value -2

E. None of the above

Type II: Discontinuous Integrands

Determine whether

$$\int_0^2 \frac{x+1}{2x-3}\,dx$$

is proper or improper, and if improper, whether it is convergent or divergent.

A. Proper

B. Improper, convergent

C. Improper, divergent

Type II: Discontinuous Integrands

Determine whether

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is proper or improper, and if improper, whether it is convergent or divergent.

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Convergent or Divergent?

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First, compute

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} [\log x]\big|_1^t, & p = 1 \\ \left[\frac{1}{1-p}x^{1-p}\right]\Big|_1^t, & p \neq 1 \end{cases}$$

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So

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} \log t, & p = 1 \\ \frac{1}{1-p}\left(t^{1-p} - 1\right), & p \neq 1 \end{cases}$$

For what values of p does $\int_1^t \frac{1}{x^p}\,dx$ approach a limit as $t \to +\infty$?

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For what values of p does $\int_1^t \frac{1}{x^p}\,dx$ approach a limit as $t \to +\infty$?

- For $p \geq 1$ the integral *diverges*

Convergent or Divergent?

For what values of p does the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} \log t, & p = 1 \\ \frac{1}{1-p}\left(t^{1-p} - 1\right), & p \neq 1 \end{cases}$$

For what values of p does $\int_1^t \frac{1}{x^p}\,dx$ approach a limit as $t \to +\infty$?

- For $p \geq 1$ the integral *diverges*
- For $p > 1$ the integral *converges*

The Gold Standard

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \begin{cases} \textit{converges}, & p > 1 \\ \textit{diverges}, & p \leq 1 \end{cases}$$

The Comparison Test for Improper Integrals

Suppose $f(x)$ and $g(x)$ are continuous functions and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Reason: If $f(x) \geq g(x) \geq 0$ then

$$\int_a^t f(x)\,dx \geq \int_a^t g(x)\,dx \geq 0.$$

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example: Determine whether $\int_1^\infty \frac{x+1}{\sqrt{x^5+1}}\,dx$ converges

Take out leading powers of x:

$$\frac{x+1}{\sqrt{x^5+1}} = \frac{x\left(1+\frac{1}{x}\right)}{x^{5/2}\sqrt{1+\frac{1}{x^5}}} = x^{-3/2}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^5}}}$$

and notice that the last factor is bounded above by 2

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

$$f(x) = x^{-3/2}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^5}}}$$

$$g(x) = \frac{x+1}{\sqrt{x^5+1}}$$

Since $f(x) \geq g(x)$ and $\int_1^\infty f(x)\,dx$ converges, $\int g(x)\,dx$ converges

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Which of the following integrals is convergent?

A. $\int_1^\infty \frac{1+e^{-x}}{x^2+4}\,dx$

B. $\int_1^\infty \frac{4}{\sqrt{x}}\,dx$

C. $\int_{-\infty}^0 \frac{3x}{x^2+4}\,dx$

D. $\int_0^\infty \frac{1}{1+x}\,dx$

Reminders

- Homework A4 on Numerical Integration is due Wednesday at 11:58 PM
- Quiz 3 on Thursday will cover sections 7.4 (partial fractions) and 7.7 (Numerical Integration)
- Homework A5 on Simpson's rule and improper integrals is due Friday at 11:58 PM
- Exam I takes place next Tuesday, February 6 at 5:00 PM - see the course website for room assignments