Math 114 - Comparison Theorem, Sequences

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Unit I: A Toolbox for Integral Calculus

Lecture 1	Integration by Parts
Lecture 2	Special Trig Integrals
Lecture 3	Trig Substitution
Lecture 4	Integrating Rational Functions, Part I
Lecture 5	Integrating Rational Functions, Part II
Lecture 6	Numerical Integration, Part I
Lecture 7	Numerical Integration, Part II
Lecture 8	Improper Integrals
Lecture 9	Comparison Theorem, Sequences
Lecture 10	(Preview) Sequences by Recursion

Overview

In this lecture we develop the *comparison test* for improper integrals, a criterion for determining whether a given improper integral *converges* or *diverges*. The idea is to compare with an integral whose convergence or divergence is already known.

We'll also introduce a new kind of function, a *sequence*, that we'll be studying in Unit II. A sequence is a function from the positive integers $\mathbb{N}$ to the real numbers $\mathbb{R}$; another way to say this is that a sequence is a list of real numbers, one for each positive integer.

Review of Improper Integrals

There are two kinds of improper integrals:

- Improper integrals of type I, where one or more of the limits of integration is infinite

  Example: $\displaystyle\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx$

- Improper integrals of type II, where the integrand is not continuous at one of the endpoints or within the interval of integration

  Example: $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x}}\,dx$

We say that the improper integral *converges* if the limit defining it exists and is finite, and *diverges* otherwise.

Convergent or Divergent?

For what values of p does the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

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First, compute

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} \left[\log x\right]\Big|_1^t, & p = 1 \\ \left[\frac{1}{1-p}x^{1-p}\right]\Big|_1^t, & p \neq 1 \end{cases}$$

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So

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} \log t, & p = 1 \\ \frac{1}{1-p}\left(t^{1-p} - 1\right), & p \neq 1 \end{cases}$$

For what values of p does $\int_1^t \frac{1}{x^p}\,dx$ approach a limit as $t \to +\infty$?

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- For $p \leq 1$ the integral *diverges*

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For what values of p does $\int_1^t \frac{1}{x^p}\,dx$ approach a limit as $t \to +\infty$?

- For $p \leq 1$ the integral *diverges*
- For $p > 1$ the integral *converges*

The Gold Standard

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \begin{cases} \textit{converges}, & p > 1 \\ \textit{diverges}, & p \leq 1 \end{cases}$$

Convergent or Divergent? A Puzzler

For $x \geq 1$, $\dfrac{1+x}{x^2} \geq \dfrac{1}{x}$.
We already know that

$$\int_1^\infty \frac{1}{x}\, dx \quad \text{(yellow)}$$

diverges. Does

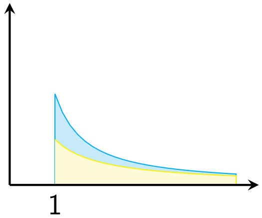
$$\int_1^\infty \frac{1+x}{x^2}\, dx \quad \text{(cyan)}$$

converge or diverge?

A. Converges

B. Diverges

C. Can't tell



Convergent or Divergent? Another Puzzler

For $x \geq 1$, $\dfrac{1}{2+x^2} \leq \dfrac{1}{x^2}$
We already know that

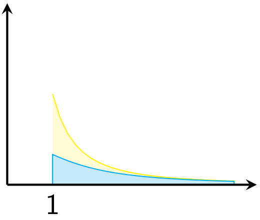
$$\int_1^\infty \frac{1}{x^2}\, dx \quad \text{(yellow)}$$

converges. Does

$$\int_1^\infty \frac{1}{2+x^2}\, dx \quad \text{(cyan)}$$

converge or diverge?

A. Converges
B. Diverges
C. Can't tell



The Comparison Test for Improper Integrals

Suppose $f(x)$ and $g(x)$ are continuous functions and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Reason: If $f(x) \geq g(x) \geq 0$ then

$$\int_a^t f(x)\,dx \geq \int_a^t g(x)\,dx \geq 0.$$

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example: Determine whether $\int_1^\infty \frac{x+1}{\sqrt{x^5+1}}\,dx$ converges

Take out leading powers of x:

$$\frac{x+1}{\sqrt{x^5+1}} = \frac{x\left(1+\frac{1}{x}\right)}{x^{5/2}\sqrt{1+\frac{1}{x^5}}} = x^{-3/2}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^5}}}$$

and notice that the last factor is bounded above by 2

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\, dx$ converges, then $\int_a^\infty g(x)\, dx$ converges.
- If $\int_a^\infty g(x)\, dx$ diverges, then $\int_a^\infty f(x)\, dx$ diverges.

$$f(x) = 2x^{-3/2}$$

$$g(x) = \frac{x+1}{\sqrt{x^5+1}}$$

Since $f(x) \geq g(x)$ and $\int_1^\infty f(x)\, dx$ converges, $\int g(x)\, dx$ converges

The Comparison Test for Improper Integrals

If f, g are continuous $f(x) \geq g(x) \geq 0$ for $x \geq a$,

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Which of the following integrals is convergent?

A. $\int_1^\infty \frac{1+e^{-x}}{x^2+4}\,dx$

B. $\int_1^\infty \frac{4}{\sqrt{x}}\,dx$

C. $\int_0^\infty \frac{3x}{x^2+4}\,dx$

D. $\int_1^\infty \frac{1}{1+x}\,dx$

Sequences

A *sequence* is an ordered, infinite list of numbers, one for each of the integers 1, 2, 3, ... The nth number in the list is denoted with a subscript, like a_n or b_n

Example The sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots \qquad a_n = 2^{-n}$$

Example The sequence

$$0, 1/2, 2/3, 3/4, \ldots, (n-1)/n, \ldots \qquad a_n = 1 - \frac{1}{n}$$

If we denote by $\mathbb{N}$ the set of positive integers $\{1, 2, 3, \ldots\}$, we can think of a sequence as a function f from $\mathbb{N}$ to $\mathbb{R}$ that assigns to each positive integer n a real number, $a_n = f(n)$

More Examples of Sequences

$a_n = \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
$a_n = \frac{(-1)^n}{n}$	$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$
$a_n = \left(\frac{2}{3}\right)^n$	$\frac{2}{3}, \frac{4}{9}, \frac{16}{27}, \frac{64}{81}, \ldots$
$a_n = n^2$	$1, 4, 9, 16, 25, 36, 49, \ldots$
a_n approximates π	$3.1, 3.14, 3.142, 3.1416, 3.14159, \ldots$

A shorthand notation for the whole infinite list of numbers is $\{a_n\}$, meaning "the set of a_n for $n = 1, 2, 3, \cdots$"

Write out the first four terms of the sequence $a_n = \dfrac{n}{n+1}$

A. $0, 1, 2, 3$
B. 1, 1/2, 1/3, 1/4
C. 1/2, 2/3, 3/4, 4/5
D. 0, 1/2, 2/3, 3/4

If It's a Function, You Can Graph It

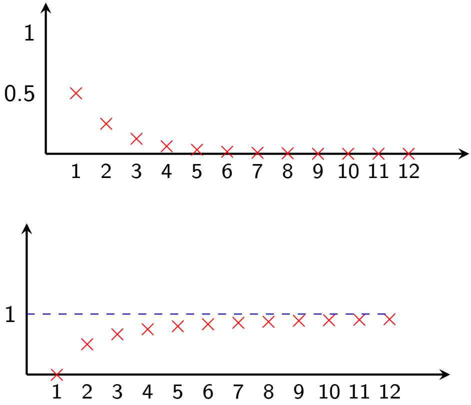
We can visualize sequences via graphs.

At right are the graphs of

$$a_n = 2^{-n}$$

and

$$a_n = 1 - \frac{1}{n}$$



If It's a Function, It Has a Limit

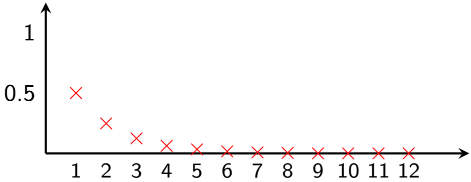
Find $\lim_{n\to\infty} 2^{-n}$

A. 0

B. 1

C. 0.5

D. Does not exist



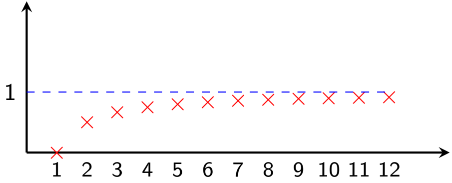
Find $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$.

A. 0

B. 1

C. 0.5

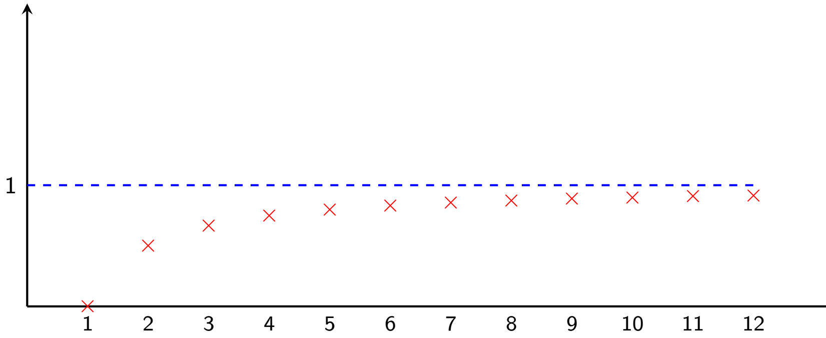
D. Does not exist



The Limit of A Sequence

We say that a sequence $\{a_n\}$ *converges* to a real number L if, given any error ε, we can make $|a_n - L| < \varepsilon$ by choosing n large enough.

Example $\lim_{n\to\infty} \dfrac{n-1}{n} = 1$



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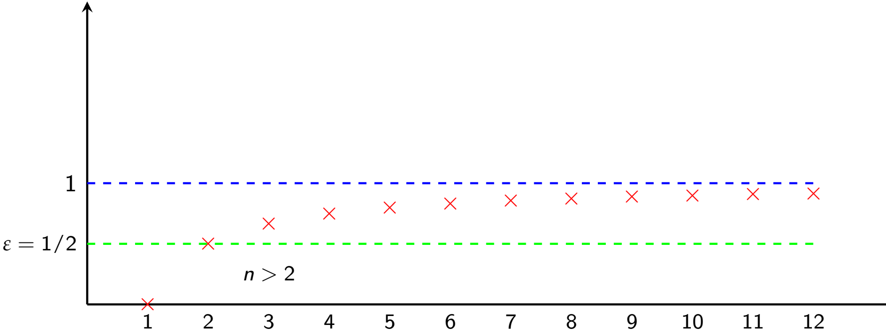

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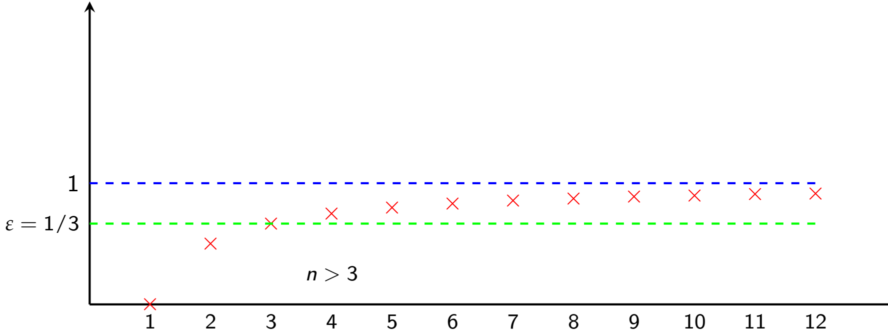

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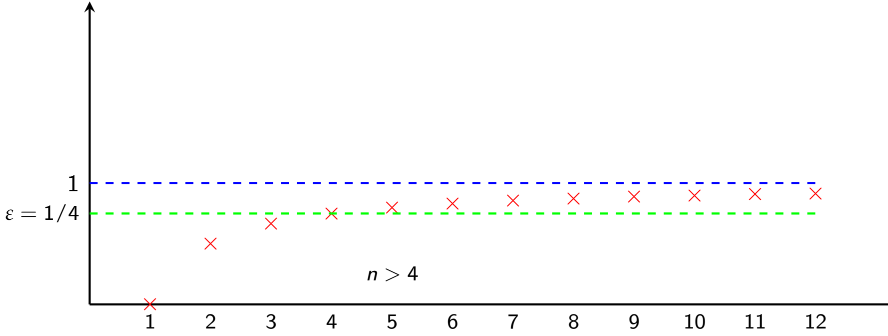

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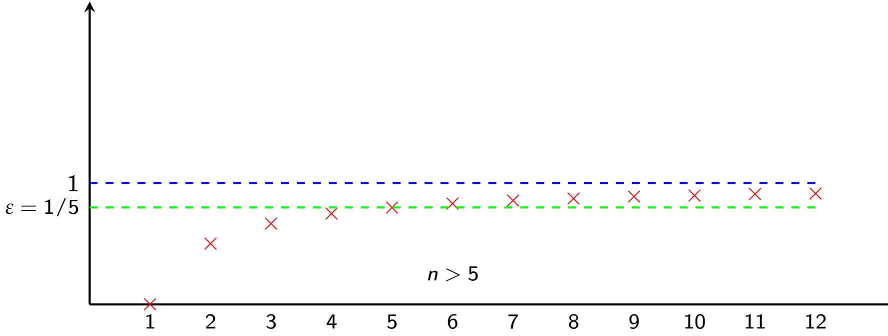

Convergent or Divergent?

If $\lim_{n\to\infty} a_n$ does not exist, we say that the sequence *diverges*.

Which of the following sequences diverges?

A. $\left\{\frac{1}{n}\right\}$

B. $\left\{\frac{(-1)^n}{n}\right\}$

C. $\{2^{-n}\}$

D. $\left\{n^2\right\}$

How Do You Find Limits of Sequences?

Theorem If f is a continuous function,with

$$\lim_{x \to \infty} f(x) = L,$$

and $\{a_n\}$ is a sequence given by $a_n = f(n)$, then

$$\lim_{n \to \infty} a_n = L.$$

Moral: All you know about finding limits (limit laws, squeeze theorem, etc.) applies to finding limits of sequences

Theorem If $\{a_n\}$ is a sequence, $\lim_{n \to \infty} a_n = L$, and f is continuous, then $\lim_{n \to \infty} f(a_n) = f(L)$.

Theorem If f is a continuous function,with

$$\lim_{x\to\infty} f(x) = L,$$

and $\{a_n\}$ is a sequence given by $a_n = f(n)$, then

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Find $\lim_{n\to\infty} 3 - e^{-n}$.

A. 0

B. 3

C. $3 - e$

D. $3 - 1/e$

E. The limit does not exist.

Theorem If $\{a_n\}$ is a sequence, $\lim_{n\to\infty} a_n = L$, and f is continuous, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Find $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$.

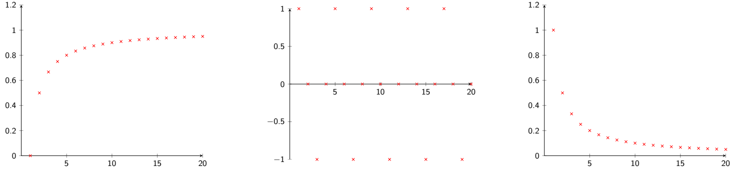
A. 1
B. 0
C. π
D. $\pi/2$
E. $\pi/4$

Increasing and Decreasing Sequences

A sequence $\{a_n\}$ is

- **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$
- **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$
- **monotonic** if it is either increasing or decreasing

Label each of the sequences shown below as increasing, decreasing, or neither.



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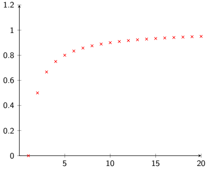
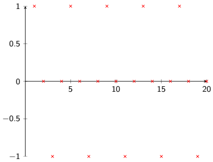
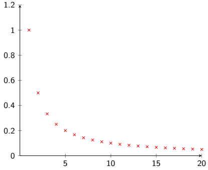






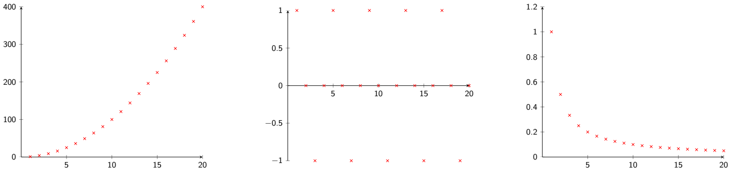
Increasing Neither Decreasing

Sequences Bounded Above and Below

A sequence $\{a_n\}$ is

- **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$
- **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$
- **bounded** if $\{a_n\}$ is bounded both above and below

Label each of the sequences below as bounded above, bounded below, bounded, or unbounded.



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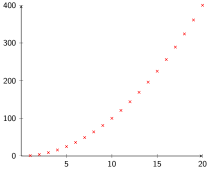
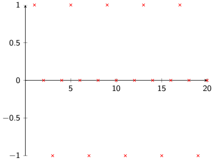
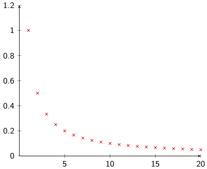






Bounded below but not above

Bounded above and below

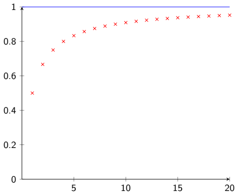
Bounded above and below

Math Into English

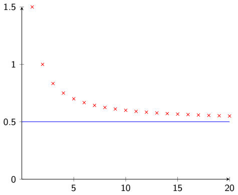
A sequence $\{a_n\}$ is

- **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$
- **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$
- **bounded** if $\{a_n\}$ is bounded both above and below

A sequence is **bounded above** if there is a "ceiling" it can't break through

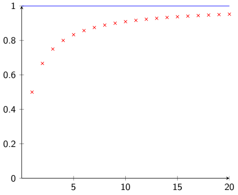


A sequence is **bounded below** if there is a "floor" it can't break through

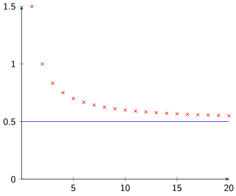


Monotonic Sequence Theorem Every bounded monotonic sequence is convergent.

This increasing, bounded above sequence "hits the ceiling" and has limit 1

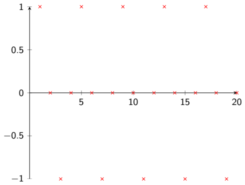


This decreasing, bounded below sequence "goes to the floor" and has limit 1/2

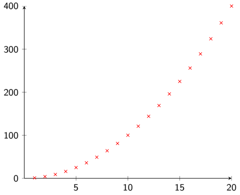


Monotonic Sequence Theorem Every bounded monotonic sequence is convergent.

This sequence $a_n = \sin(\pi n/2)$ is bounded but not monotonic. It doesn't converge.



This sequence $a_n = n^2$ is monotonic but not bounded. It doesn't converge.



Reminders

- Quiz 3 on Thursday will cover sections 7.4 (partial fractions) and 7.7 (Numerical Integration)
- For Friday, download and read the handout on **Recursive Sequences**
- Homework A5 on Simpson's rule and improper integrals is due Friday at 11:58 PM
- On Monday in class we'll review for Exam I. There is also a **review session** on Monday, February 5, 6:00-8:00 PM, in FB 200 open to all students in MA 114.
- Exam I takes place next Tuesday, February 6 at 5:00 PM - see the course website for room assignments