MA 213 — Calculus III | Fall 2017
Exam 3 | November 15, 2017

Name: KEY

Section: ____________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

Exam Scores

Do not write in the table below

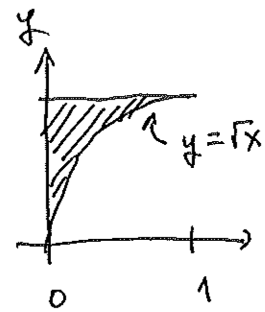
Question	Score	Total
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Free Response. Always show your work!

1. (10 points) Change the order of integration in the following iterated integral:

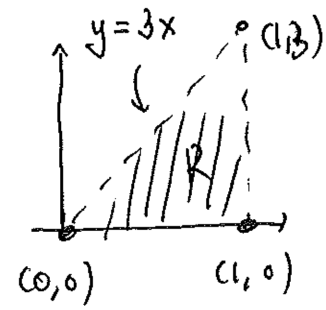
$$\int_0^1 \int_{\sqrt{x}}^1 e^{x^3 y}\, dy\, dx.$$

Do not evaluate the integral.



$$\int_0^1 \left(\int_0^{y^2} e^{x^3 y}\, dx \right) dy$$

2. (10 points) Find the average value of $f(x, y) = xy$ over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.



The area of the triangle is

$$\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$$

The average value is

$$\frac{2}{3} \iint_R xy\, dA = \frac{2}{3} \int_0^1 \left(\int_0^{3x} xy\, dy \right) dx$$

$$= \frac{2}{3} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{3x} dx$$

$$= \frac{2}{3} \int_0^1 \frac{9x^3}{2}\, dx$$

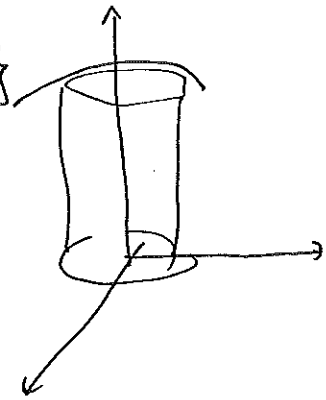
$$= \frac{2}{3} \cdot \frac{9}{2} \cdot \left[\frac{x^4}{4} \right]_0^1$$

$$\cancel{= \frac{3}{2} \cdot \frac{1}{4} =}$$

$$= \boxed{\frac{3}{4}}$$

3. (10 points) Use polar coordinates to find the volume of the solid E inside the cylinder $x^2 + y^2 = 1$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$, and above the xy-plane.

$$E = \{(x,y,z) : x^2+y^2 \le 1,\ 0 \le z \le \sqrt{64-4x^2-4y^2}\}$$



solve for z in $4x^2+4y^2+z^2=64$

In polar coordinates:

$$E = \{(r,\theta,z) : r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le \sqrt{64-4r^2}\}$$

The volume is

$$\iint_D z\, dA \quad \text{where} \quad D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

so

$$V = \int_0^{2\pi} \left(\int_0^1 \sqrt{64-4r^2}\, r\,dr \right) d\theta$$

$u = 64-4r^2$

$du = -8r\,dr$

r	u
0	64
1	60

$$= \int_0^{2\pi} \left(\int_{60}^{64} \sqrt{u}\left(\tfrac{1}{8}\right) du \right) d\theta$$

$$= \frac{2\pi}{8} \left(\frac{2}{3} u^{3/2} \Big|_{60}^{64} \right)$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \left(64^{3/2} - 60^{3/2} \right)$$

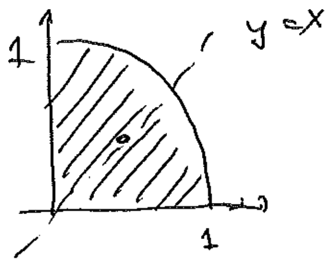
$$= \frac{\pi}{6} \left(8 \cdot 64 - 60 \cdot \sqrt{60} \right)$$

Free Response. Always show your work!

4. (10 points) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass, if the density is given by $\rho(x, y) = x + y$.

Note: You may need to use the formulas

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2}, \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}.$$



1) By symmetry, the center of mass lies along the line $y = x$

2) The total mass of the lamina is

$$\iint_R (x+y)\, dA = \int_0^{\pi/2} \int_0^1 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}(\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{1}{3}\left[\sin\theta - \cos\theta\right]\Big|_0^{\pi/2} = \frac{1}{3}[1+1] = \boxed{\frac{2}{3}}$$

3) The moment about the x axis is

$$M_x = \iint y(x+y)\, dA = \int_0^{\pi/2} \int_0^1 r^2 (\cos\theta \sin\theta + \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} (\cos\theta \sin\theta + \sin^2\theta)\, d\theta$$

① : use $u = \sin\theta$, $du = \cos\theta\, d\theta$

② : use $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$= \frac{1}{4}\left(\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2} + \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]\Big|_0^{\pi/2} \right)$$

$$= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \left(\frac{\pi}{4}\right) = \frac{1}{8} + \frac{\pi}{16}$$

$$\therefore \quad \bar{y} = \frac{M_x}{M} = \frac{3}{2} \cdot \left(\frac{1}{8} + \frac{\pi}{16}\right) = \frac{3}{16} + \frac{3\pi}{32}$$

Free Response. Always show your work!

5. (10 points) Find the surface area of the part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$.

$z = \frac{1}{2} - 3x - 2y$ $\quad \frac{\partial z}{\partial x} = -3$ $\quad \frac{\partial z}{\partial y} = -2$ $\quad \therefore \ dS = \sqrt{1 + (-3)^2 + (-2)^2}\ dx\,dy$

$= \sqrt{14}$

$S = \iint_D \sqrt{14}\ dA$ The cylinder has a circular cross section of area 25π so

$\boxed{S = 25\sqrt{14}\,\pi}$

6. (10 points) Evaluate

$$\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz.$$

"Inside out"

$$\int_0^2 \left(\int_0^{z^2} \left(\int_0^{y-z} (2x-y)\ dx \right) dy \right) dz = \int_0^2 \left(\int_0^{z^2} \left((x^2 - xy)\Big|_0^{y-z} \right) dy \right) dz$$

$$= \int_0^2 \int_0^{z^2} (y-z)^2 - y(y-z)\ dy\, dz$$

$$= \int_0^2 \int_0^{z^2} (z^2 - yz)\ dy\, dz$$

$$= \int_0^2 \left[z^2 y - \frac{zy^2}{2} \right]\Big|_0^{z^2} dz$$

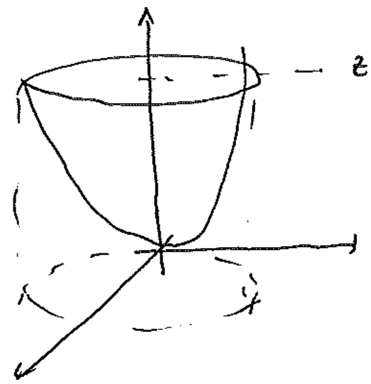
$$= \int_0^2 \left(z^4 - \frac{1}{2} z^5 \right) dz$$

$$= \left[\frac{z^5}{5} - \frac{z^6}{12} \right]\Big|_0^2$$

$$= \frac{32}{5} - \frac{64}{12} = \frac{32}{5} - \frac{16}{3} = \frac{32\cdot 3 - 16\cdot 5}{15} = \boxed{\frac{16}{15}}$$

Free Response. Always show your work!

7. (10 points) Compute the volume of the solid bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a\,(a > 0)$.



The surfaces intersect at

$4x^2 + 4y^2 = \sqrt{a}$

or $x^2 + y^2 = \frac{a}{4}$ which is a circle of radius $\sqrt{a}/2$.

The region is

$\{(r,\theta,z) : \; 0 \le r \le \frac{\sqrt{a}}{2}, \; 0 \le \theta \le 2\pi, \; 0 \le z \le 4r^2\}$

$z = 4(x^2+y^2) = 4r^2$

so

$$V = \iint_R z \, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{a}/2} 4r^2 \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\cancel{4} \cdot \frac{r^4}{\cancel{4}} \right]_0^{\sqrt{a}/2} d\theta$$

$$\cancel{= \frac{2\pi}{5} \cdot \left(\frac{\sqrt{a}}{2}\right)^8}$$

$$= 2\pi \, \frac{a^2}{2^4} = \boxed{\frac{\pi a^2}{8}}$$

8. (10 points) Find the spherical coordinates (ρ, θ, ϕ) of the point P whose rectangular coordinates are $(-1, 1, -\sqrt{2})$.

$\rho = \sqrt{1+1+2} = 2$

$\cos\phi = \frac{z}{\rho} = -\frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{3\pi}{4}$

$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = \frac{3\pi}{4}$

$\therefore (\rho, \theta, \phi) = (2, \frac{3\pi}{4}, \frac{3\pi}{4})$

9. (10 points) In spherical coordinates, the equation $\rho = 2\sin\phi\sin\theta$ represents a sphere. Find the center C and the radius R of this sphere.

$\sin\phi = \frac{r}{\rho}$ $\sin\theta = \frac{y}{r}$ where $r = \sqrt{x^2+y^2}$

$\therefore \rho = 2\frac{\cancel{r}}{\rho} \cdot \frac{y}{\cancel{r}} = 2y/\rho$

$\therefore \rho^2 = 2y$

$\therefore x^2+y^2+z^2 = 2y$

$\therefore x^2+(y^2-2y+1)+z^2 = 1$ (complete the square!)

$\therefore x^2+(y-1)^2+z^2 = 1$

$\therefore$ $\boxed{\text{Radius } 1, \text{ center } (0,1,0)}$

10. (10 points) (a) Describe in spherical coordinates the region E that lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

$2 \le \rho \le 3, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$

(b) Express $x^2 + y^2$ in spherical coordinates (ρ, θ, ϕ).

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$\therefore \; x^2 + y^2 = \rho^2 \sin^2\phi \,(\cos^2\theta + \sin^2\theta)$$
$$= \rho^2 \sin^2\phi$$

(c) Write the triple integral

$$\iiint_E (x^2 + y^2)\, dV$$

as an iterated integral using spherical coordinates. Do not evaluate the integral.

$E = \{(\rho, \theta, \phi) : 2 \le \rho \le 3, \; 0 \le \phi \le \pi, \; 0 \le \theta \le 2\pi\}$

$$\iiint_E (x^2+y^2)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_2^{3} (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$