MA 213 — Calculus III Spring 2018
Exam 4 May 2, 2018

Name: KEY

Section: ____________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

Exam Scores

Do not write in the table below

Question	Score	Total
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Free Response. Show your work!

1. (10 points) The plane $8x + by + cz = d$ passes through the point $(3, 5, -1)$ and contains the line

$$x = 4 - t, \quad y = 2t - 1, \quad z = -3t.$$

Find $b, c,$ and d.

(1) $(3,5,-1)$: $24 + 5b - c = d$

The line at $t=0$ lies at the point $(4,-1,0)$

(2) $32 - b + 0 = d$

The normal $(8,b,c)$ is orthogonal to $\langle -1, 2, -3\rangle$

(3) $-8 + 2b - 3c = 0$

Subtract (1) from (2): $8 - 6b + c = 0$

Use (3) $\times 3$ $-24 + 6b - 9c = 0$

$-16 \quad - 8c = 0$

$c = -2$

Use (3) to find b:

$-8 + 2b + 6 = 0$

$2b = 2$

$b = 1$

Use (1) to find d:

$24 + 5 \cdot 1 + 2 = d$

$d = 31$

$\boxed{8x + y - 2z = 31}$

$b = 1$

$c = -2$

$d = 31$

2. (10 points) Find the cosine of the angle θ between the vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$. Is θ acute or obtuse?

$\vec{a} \cdot \vec{b} = 0 + (-3)(2) + (1)(-1) = -7$

$|\vec{a}| = \sqrt{16 + 9 + 1} = \sqrt{26}$

$|\vec{b}| = \sqrt{4 + 1} = \sqrt{5}$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-7}{\sqrt{5}\sqrt{-26}}$

Since $\cos\theta < 0$, $\theta > \frac{\pi}{2}$ (obtuse)

Free Response. Show your work!

3. (10 points) Find the parametric equations for the tangent line to the curve

$$x = t, \quad y = e^{-t}, \quad z = 2t - t^2$$

at the point $(0, 1, 0)$. (corresponding to $t=0$)

$x'(t) = 1 \qquad y'(t) = -e^{-t} \qquad z'(t) = 2-2t$

$x'(0) = 1 \qquad y'(0) = -1 \qquad z'(0) = 2$

$\vec{r}(0) = \langle 0,1,0 \rangle \qquad \vec{r}'(0) = \langle 1,-1,2 \rangle$

Tgt line

$\vec{r}(t) = \langle 0,1,0 \rangle + t\langle 1,-1,2 \rangle$

or: $x(t) = t$

$y(t) = 1-t$

$z(t) = 2t$

4. (10 points) Find the partial derivatives $f_x(3, -4)$ and $f_y(3, -4)$ for the function

$$f(x, y) = \frac{2x + y}{3x + 2y}.$$

Easy way: ① $f(x,-4) = \dfrac{2x-4}{3x-8}$

$\dfrac{\partial f}{\partial x}(x,-4) = \dfrac{2(3x-8)-(2x-4)(3)}{(3x-8)^2}$

$\dfrac{\partial f}{\partial x}(3,-4) = \dfrac{2\cdot(1)-(2)(3)}{1^2} = \boxed{-4}$

② $f(3,y) = \dfrac{6+y}{9+2y}$

$\dfrac{\partial f}{\partial y}(3,y) = \dfrac{(9+2y)-2(6+y)}{(9+2y)^2}$

$\dfrac{\partial f}{\partial x}(3,-4) = \dfrac{1-2(2)}{1^2} = \boxed{-3}$

Free Response. Show your work!

5. (10 points) Find the directional derivative of $f(x,y)=\sqrt{xy}$ at the point $P(2,8)$ in the direction of the point $Q(5,4)$.

$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x}}\cdot\sqrt{y}$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}\sqrt{x}$

$\frac{\partial f}{\partial x}(2,8) = \frac{1}{2\sqrt{2}}\sqrt{8} = 1$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{8}}\cdot\sqrt{2} = \frac{1}{4}$

$\vec{PQ} = \langle 3,-4\rangle$ so a unit vector in the $\vec{PQ}$ direction is $\vec{u}$ $\langle \frac{3}{5}, \frac{-4}{5}\rangle$

$\therefore\ (D_{\vec{u}}f)(2,8) = \langle 1, \frac{1}{4}\rangle\cdot\langle\frac{3}{5},\frac{-4}{5}\rangle = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$

6. (10 points) Use Lagrange multipliers to find the extreme values of $f(x,y,z)=2x+2y+z$ on the sphere $x^2+y^2+z^2=9$. You must use Lagrange multipliers: no other method will be accepted.

$\nabla f = \langle 2,2,1\rangle$ $\qquad$ $\nabla g = \langle 2x, 2y, 2z\rangle$

$\nabla f = \lambda\nabla g \Rightarrow$ $\quad 2 = 2\lambda x$, $\quad 2 = 2\lambda y$, $\quad 1 = 2\lambda z$

$\lambda \neq 0$ otherwise there are no solutions. Solving for λ we get

$\frac{1}{\lambda} = x = y = 2z$

Put $y=x$, $z=\frac{x}{2}$ into eqn for sphere:

$x^2 + x^2 + \frac{1}{4}x^2 = 9$

$\frac{9x^2}{4} = 9$

$x = \pm 2$

$\therefore$ Extrema occur at

$(+2,+2,1)$ and $(-2,-2,-1)$.

$f(2,2,1) = 4+4+1 = 9$ max

$f(-2,-2,-1) = -4-4-1 = -9$ min

Free Response. Show your work!

7. (10 points) Evaluate

$$\int_C x^2 y\, ds,$$

where C is the curve

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad (0 \le t \le \pi/2).$$

$x'(t) = -\sin t \quad y'(t) = \cos t \quad z'(t) = 1$

$ds = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\therefore \int_C x^2 y \, ds = \int_0^{\pi/2} \cos^2 t \sin t \sqrt{2}\, dt = \sqrt{2} \int_0^{\pi/2} \cos^2 t \sin t \, dt$

$(u = \cos t) \qquad = \sqrt{2} \int_0^1 u^2 du$

$= \sqrt{2} \cdot \frac{1}{3} = \boxed{\frac{\sqrt{2}}{3}}$

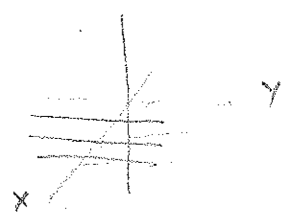
8. (10 points) Let E be the solid

$$E = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1,\ x \ge 0,\ z \ge 0\}.$$

Write the triple integral

$$\iiint_E z e^{x^2+y^2+z^2}\, dV$$

in spherical coordinates. Do not evaluate the integral.



The region E is described in spherical coordinates by

$0 \le \rho \le 1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \pi/2$

Substitute $x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$

$\therefore \iiint z e^{x^2+y^2+z^2} dV = \boxed{\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^1 \rho \cos\phi \, e^{\rho^2} (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi}$

Free Response. Show your work!

9. (10 points) Let

$$\mathbf{F}(x,y) = \langle 2xe^{-y},\, 2y - x^2e^{-y} \rangle.$$

Find a potential function $f(x,y)$ for $\mathbf{F}(x,y)$ and evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any path from $(1,0)$ to $(3,0)$.

$\frac{\partial f}{\partial x} = 2xe^{-y}$ $\qquad$ $\frac{\partial f}{\partial y} = 2y - x^2e^{-y}$

$\Downarrow$ $\qquad$ $\Downarrow$

$f(x,y) = x^2e^{-y} + C(y)$ $\qquad$ $-x^2e^{-y} + C'(y) = 2y - x^2e^{-y}$

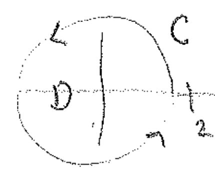
$\therefore C'(y) = 2y$

$\therefore C(y) = y^2 + C$

$\therefore f(x,y) = x^2e^{-y} + y^2 + C$

$\therefore \int_C \vec{F} \cdot d\vec{r} = f(3,0) - f(1,0) = 9 - 1 = \boxed{8}$

10. (10 points) Use Green's theorem to evaluate



$$\oint_C y^3\,dx - x^3\,dy,$$

where C is the positively oriented circle $x^2 + y^2 = 4$. You must use Green's theorem: no other method will be accepted.

$P(x,y) = y^3$ $\qquad$ $Q(x,y) = -x^3$ $\qquad$ $\frac{\partial Q}{\partial x} = -3x^2$ $\quad$ $\frac{\partial P}{\partial y} = 3y^2$

$\oint_C y^3\,dx - x^3\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$= \iint_D -3(x^2+y^2)\, dA$

$= \int_0^{2\pi} \int_0^2 -3r^2 \, r\,dr\,d\theta = 2\pi \int_0^2 -3r^3\,dr = -6\pi \cdot \frac{r^4}{4} \Big|_0^2$

$= \boxed{-24\pi}$