Lecture 14: Addendum

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These notes complete the example I didn't finish in class. Remember that we stated the following theorem

Theorem If $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous at (a,b), then f is differentiable at (a,b).

Recall that f is differentiable at (a,b) if

$$f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon \Delta x + \varepsilon \Delta y$$

where $\varepsilon \to 0$ as $(x,y) \to (a,b)$.

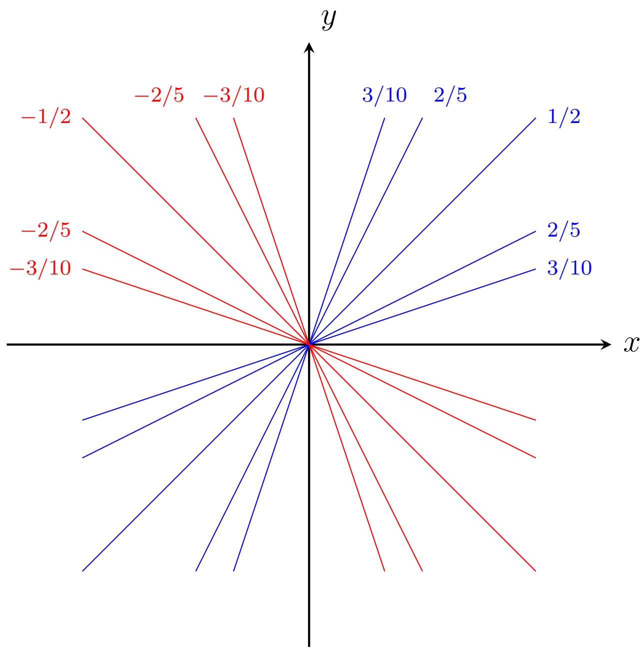
We want to provide an example of a function for which $f_x(a,b)$ and $f_y(a,b)$ both exist, but the derivatives $f_x(x,y)$ and $f_y(x,y)$ are not continuous at (a,b), and f is not differentiable at (a,b). The example is:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ \\ 0 & (x,y) = (0,0) \end{cases} \qquad (a,b) = (0,0).$$

The graph of this function looks very strange at $(0,0)$ (see the lecture slide for the graph). We can understand this "strangeness" a bit more by considering how $f(x,y)$ behaves as $(x,y) \to (0,0)$ along the line $y = mx$. We get

$$f(x,mx) = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

so that f is actually *constant* along the line, but the constant value is *different* depending on m, the slope of the line. The level curves $f(x,y) = k$ look like this:



Clearly, something *very* weird is going on at $(0,0)$!

For $(x,y) \neq (0,0)$ we can compute the partials of $f(x,y)$ using the quotient rule. We get

$$f_x(x,y) = \frac{y(y^2 - x^2)}{(x^2+y^2)^2}, \quad f_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

The behavior of these derivatives as $(x,y) \to (0,0)$ is even spookier than the behavior of $f(x,y)$. If we try the same trick of following the derivative to zero by along lines we get

$$f_x(x,mx) = \frac{m(m^2-1)}{x(1+m^2)^2}, \quad f_y(x,mx) = \frac{(1-m^2)}{x(1+m^2)^2}$$

so that, as $x \to 0$ along any line, the partial derivatives increase without bound (since $1/x \to \pm\infty$ as $x \to 0$)!

More strangely yet, we can use the definition of derivative to show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$. First, note that $f(h,0) = f(0,h) = 0$. Next, compute

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

and

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

The moral of this story is that even rational functions of two variables can be non-differentiable (and not even continuous) at certain points!