# Math 213 - Directional Derivatives and the Gradient

Peter A. Perry

University of Kentucky

October 2, 2019

# Reminders

- Homework B3 is due tonight!
- You have Quiz #4 on 14.1 and 14.3 in recitation tomorrow
- Homework B4 is due on Friday

# Unit II: Functions of Several Variables

# Learning Goals

- Understand what a *directional derivative* of a function of two and three variables is, and how to compute it
- Understand what the *gradient vector* $\nabla f$ is, and how it's related to directional derivatives
- Understand that the gradient vector:
  - Points in the direction of maximum change of $f$
  - Has magnitude equal to that maximal rate of change
  - Is perpendicular to level curves (two variables) or level surfaces (three variables)

# Review

**The Chain Rule, 2 Variables (Case 1)** If

$$z = f(x, y),\ x = g(t),\ y = h(t),$$

then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that

$$f_x(1,3) = -2, \quad f_y(1,3) = 4,$$
$$x(t) = t, \quad y(t) = 3t.$$

What is the derivative of $f(x(t), y(t))$ at $t = 1$?

# Review

**The Chain Rule, 2 Variables (Case 1)** If

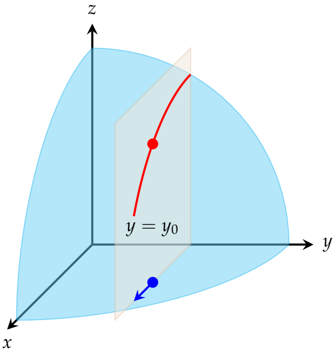
$$z = f(x,y),\ x = g(t),\ y = h(t),$$

then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

If $x(t) = x_0 + at$, $y(t) = y_0 + bt$, what is $(d/dt)f(x(t), y(t))$ at $t = 0$?

# Review

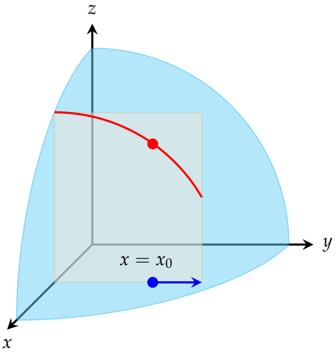



Suppose $f(x,y)$ is a function of two variables, and $(x_0,y_0)$ is a point in its domain. The partial derivatives of $f$ with respect to $x$ and $y$ are given by:

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

(change in direction of $\langle 1,0\rangle$)

# Review




Suppose $f(x,y)$ is a function of two variables, and $(x_0, y_0)$ is a point in its domain. The partial derivatives of $f$ with respect to $x$ and $y$ are given by:

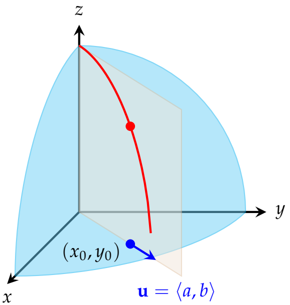
$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

(change in direction of $\langle 1,0\rangle$)

$$f_y(x_0,,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

(change in direction of $\langle 0,1\rangle$)

# The Directional Derivative




Suppose $f(x,y)$ is a function of two variables, $(x_0, y_0)$ is a point in its domain, and $\mathbf{u} = \langle a, b \rangle$ is a unit vector.

The directional derivative of $f$ in the direction $u = \langle a, b \rangle$ at $(x_0, y_0)$ is

$$D_{\mathbf{u}} f(x_0, y_0) =$$

$$\lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

# Computing the Directional Derivative

Remember that

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If $\mathbf{u} = \langle a, b \rangle$, this is the same as the derivative of $f(x_0 + at, y_0 + bt)$ at $t = 0$.

We can compute this by the chain rule and get

$$D_{\mathbf{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

1. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $\mathbf{u} = \langle 1/2, \sqrt{3}/2 \rangle$
2. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $\mathbf{u} = (-5/13)\mathbf{i} + (12/13)\mathbf{j}$

# The Gradient Vector

We can look at the formula

$$D_{\mathbf{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

in a new way by introducing the *gradient of $f$ at* $(x_0, y_0)$: if

$$(\nabla f)(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

then

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f \cdot \mathbf{u}$$

1. Find the gradient vector for $f(x, y) = x/y$ at $(2, 1)$
2. Find the directional derivative of $f$ at $(2, 1)$ in the direction $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

# Maximum Rate of Change

Remember that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos(\theta)$$

where $\theta$ is the angle between **a** and **b**.

The dot product has its maximum value when the vector **a** points in the same direction as **b**.

So, the directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ has its maximum when **u** points in the same direction as $\nabla f(x_0, y_0)$. In this direction, $D_{\mathbf{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)|$

1. Find the maximum rate of change of $f(x, y) = xe^{xy}$ at $(0, 2)$ and find the direction where it occurs.

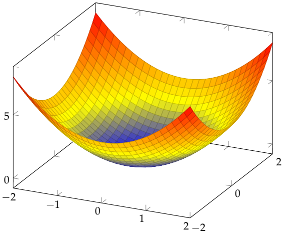
# Sneak Preview: The Gradient Vector Field

If $f(x,y)$ is a function two variables, the *gradient vector field*

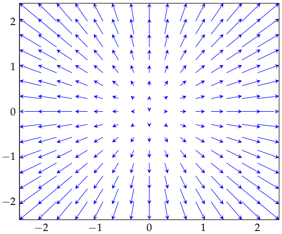
$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}(x,y)\mathbf{j}$$

moves in the direction of greatest change of $f$

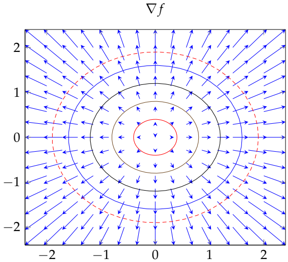
$f(x) = x^2 + y^2$




$\nabla f$



# The Gradient and Level Curves




The dot product of two vectors **a** and **b** is zero when **a** and **b** are perpendicular.

The directional derivative $D_{\mathbf{u}} f$ must be zero when **u** points along a level curve.

So, the gradient of a function $f$ must be perpendicular to the level curves of $f$

# Summary

- The *gradient* of a function $f(x, y)$ at $(x, y) = (x_0, y_0)$ is the vector

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

- If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then the directional derivative of $f$ at $(x_0, y_0)$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

- The gradient $\nabla f(x_0, y_0)$ points in the direction of greatest change of $f$ at $(x_0, y_0)$. The magnitude of the gradient is equal to the greatest change.
- The gradient $\nabla f(x_0, y_0)$ is perpendicular to the level curve of $f$ passing through $(x_0, y_0)$

# Directional Derivatives of $f(x, y, z)$

The directional derivative of $f(x, y, z)$ at $(x_0, y_0, z_0)$ in the direction $\mathbf{u} = \langle a, b, c \rangle$ is given by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

To compute it, we introduce the gradient vector

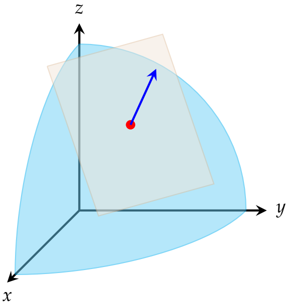
$$\nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}$$

Then, if $\mathbf{u} = \langle a, b, c \rangle$ is a unit vector,

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u}.$$

1. Using the gradient vector, find the directional derivative of $f(x, y, z) = y^2 e^{xyz}$ at $(0, 1, -1)$ in the direction $\mathbf{u} = \frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$.
2. Find the maximum rate of change of $f(xy, z) = x \ln(yz)$ at $(1, 2, 1/2)$ and find the direction in which it occurs.

# Tangent Planes to Level Surfaces




The gradient of a function of two variables is perpendicular to level curves of that function.

The gradient of a function of three variables is perpendicular to level surfaces of that function.

This means that $\nabla f(x_0, y_0, z_0)$ is normal to the tangent plane to $f$ at $(x_0, y_0, z_0)$

1. Find the equations of the tangent plane and the normal line to the surface $x = y^2 + z^2 + 1$ at $(3, 1, -1)$.
2. Are there any points on the hyperboloid $x^2 - y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

# Summary

- We defined directional derivative $D_{\mathbf{u}}f(a,b)$ of a function of two variables - the rate of change of $f$ in the direction $\mathbf{u}$ at $(a,b)$
- We introduced the gradient vector $\nabla f$ and found how to compute the directional derivative using the gradient vector: $D_{\mathbf{u}}f(a,b) = \mathbf{u} \cdot \nabla f(a,b)$
- We learned the following properties of $\nabla f$:
  - $\nabla f(a,b)$ points in the direction of greatest increase of $f$ at $(a,b)$
  - $|\nabla f(a,b)|$ is the maximum rate of change of $f$ at $(a,b)$
  - $\nabla f$ is perpendicular to the level curves of $f$ (two variable) or level surfaces of $f$ (three variables)