Math 213 - Triple Integrals in Cylindrical Coordinates

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Reminders

- Homework C2 on section 15.6 (triple integrals) is due **tonight**!
- Homework C3 on section 15.7 (triple integrals in cylindrical coordinates) is due Wednesday night.
- Quiz #7 on sections 15.3 and 15.6 is on Thursday of this week.
- Homework C4 on section 15.8 (triple integrals in spherical coordinates) is due Friday night.
Unit III: Multiple Integrals, Vector Fields

Double Integrals in Polar Coordinates
Triple Integrals (Part I)
Triple Integrals (Part II)
**Triple Integrals in Cylindrical Coordinates**
Triple Integrals in Spherical Coordinates
Change of Variables, Part I
Change of Variables, Part II

Vector Fields
Line Integrals (Scalar functions)
Line Integrals (Vector functions)

Exam III Review
Goals of the Day

- Understand how to describe regions in $xyz$ space with cylindrical coordinates
- Understand how to set up triple integrals as iterated integrals in cylindrical coordinates
Cylindrical Coordinates

Polar coordinates \((r, \theta)\) locate points in the \(xy\) plane
Cylindrical Coordinates

Polar coordinates \((r, \theta)\) locate points in the \(xy\) plane

Add the \(z\)-coordinate to polar coordinates and you get *cylindrical coordinates*
Cylindrical Coordinates

Recall conversions to and from polar coordinates:

\[ r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x} \]
\[ x = r \cos \theta, \quad y = r \sin \theta \]
Cylindrical Coordinates

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\[ x = r \cos \theta, \quad y = r \sin \theta \]

1. Find the cylindrical coordinates of the point \((-1, 1, 1)\)
2. Find the cylindrical coordinates of the point \((-2, 2\sqrt{3}, 3)\)
3. Find the rectangular coordinates of the point \((4, \pi/3, -2)\)
Equations and Regions in Cylindrical Coordinates

1. Identify the polar curve \( r = 2 \sin \theta \)
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Equations and Regions in Cylindrical Coordinates

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2. Identify the surface \( r = 2 \sin \theta \)
3. Write the equation
   \[
   2x^2 + 2y^2 - z^2 = 4
   \]
   in cylindrical coordinates
Equations and Regions in Cylindrical Coordinates

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4. Sketch the solid described by the inequalities \( 0 \leq \theta \leq \pi/2, \ r \leq z \leq 2 \)
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Triple Integrals in Cylindrical Coordinates

In polar coordinates

\[ dA = r \, dr \, d\theta \]

So, in cylindrical coordinates,

\[ dV = r \, dr \, d\theta \, dz = r \, dz \, dr \, d\theta \]

If \( E \) is the region

\[ E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \]

then

\[ \int\int\int_E f(x, y, z) \, dV = \int\int_D \left( \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) \, dA \]

If we can describe \( D \) in polar coordinates:

\[ D = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\} \]

then we can evaluate

\[ \int\int\int_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \]
Step by Step

\[ \int \int \int_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \]

This formula summarizes a multi-step process. If

\[ E = \{(x, y, z) : (x, y) \in D, \ u_1(x, y) \leq z \leq u_2(x, y)\} \]

then, to use the formula:

1. Substitute \( x = r \cos \theta, y = r \sin \theta \) into \( u_1 \) and \( u_2 \) to find the limits of the inmost integral

2. Substitute \( x = r \cos \theta, y = r \sin \theta \) into the formula for \( f(x, y, z) \) to rewrite \( f \) as a function of \( r, \theta, \) and \( z \)

3. After making these substitutions, evaluate the triple iterated integral
Learning Goals

Triple Integrals in Cylindrical Coordinates

\[ \iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \]

1. Find \( \iiint_E z \, dV \) where \( E \) is enclosed by the paraboloid
\[ z = x^2 + y^2 \]
and the plane \( z = 4 \)
Triple Integrals in Cylindrical Coordinates

\[
\iiint_E f(x, y, z) \, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta
\]

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   \[ z = x^2 + y^2 \]
   and the plane \( z = 4 \)

2. Find \( \iiint_E (x - y) \, dV \) if \( E \) is the solid which lies between the cylinders
   \[ x^2 + y^2 = 1, \quad x^2 + y^2 = 16, \]
   above the xy plane, and below the plane \( z = y + 4 \).