Math 213 - Triple Integrals in Spherical Coordinates

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October 30, 2019

Reminders

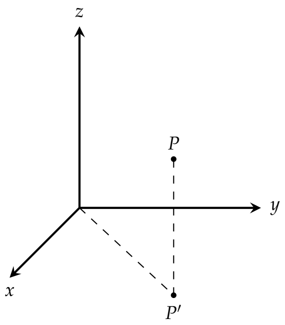
- Homework C3 on section 15.7 (triple integrals in cyindrical coordinates) is due tonight
- Quiz #7 on sections 15.3 and 15.6 takes place tomorrow in recitation
- Homework C4 on section 15.8 (triple integrals in spherical coordinates) is due Friday

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

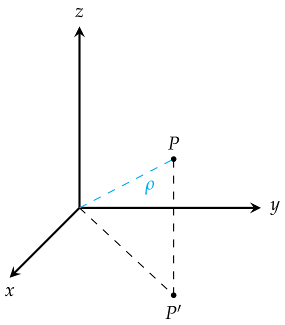
- Know how to locate points and describe regions in spherical coordinates
- Know how to evaluate triple integrals in spherical coordinates

Spherical Coordinates



The spherical coordinates (ρ, θ, ϕ) of a point P in three-dimensional space with projection P' on the xy-plane are:

Spherical Coordinates



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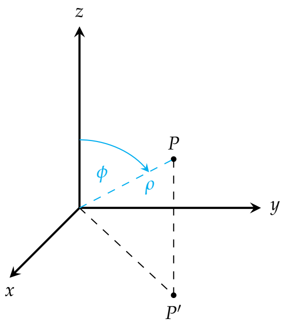
- $\rho = \sqrt{x^2 + y^2 + z^2}$, the distance $|\overrightarrow{OP}|$

Spherical Coordinates



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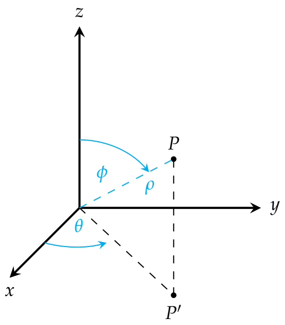
- ϕ, the angle that the vector $\overrightarrow{OP}$ makes with the z-axis

Spherical Coordinates



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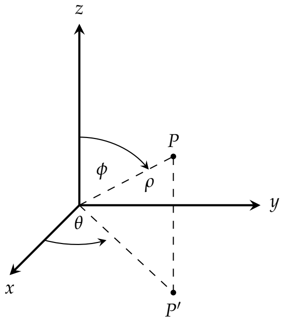
- ϕ, the angle that the vector $\overrightarrow{OP}$ makes with the z-axis
- θ, the angle that the vector $\overrightarrow{OP'}$ makes with the x-axis

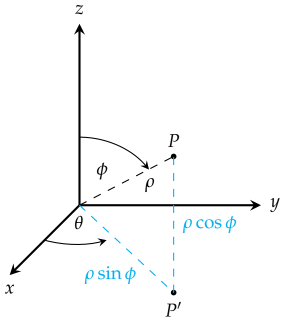
Cartesian to Spherical and Back Again

Going over:



$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\cos\phi = \frac{z}{\rho}$$

Cartesian to Spherical and Back Again



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Coming back:

$$x = \rho \sin\phi \cos\theta$$
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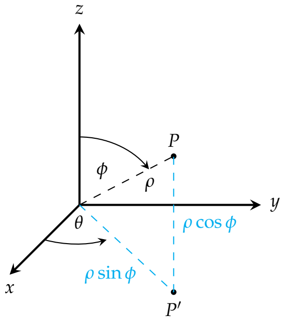

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Coming back:

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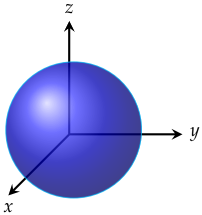
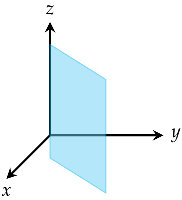
1. Find the spherical coordinates of the point $(1, \sqrt{3}, 4)$
2. Find the cartesian coordinates of the point $(4, \pi/4, \pi/2)$

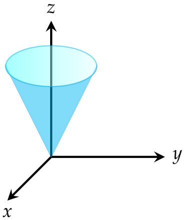
Regions in Spherical Coordinates

Match each of the following surfaces with its graph in xyz space

1. $\theta = c$
2. $\rho = 5$
3. $\phi = c, \quad 0 < c < \pi/2$





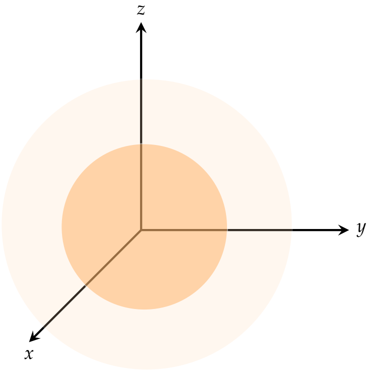


A Spherical Wedge

The region

$$E = \{(\rho, \theta, \phi) : a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

is a *spherical wedge*. What does it look like?



- $a \leq \rho \leq b$ means the shape lies between spheres of radius a and b

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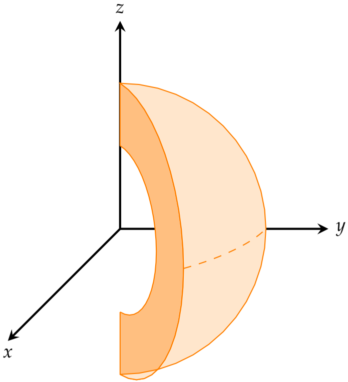

- $a \leq \rho \leq b$ means the shape lies between spheres of radius a and b
- $\alpha \leq \theta \leq \beta$ restricts the shape to a wedge-shaped region over the xy plane

A Spherical Wedge

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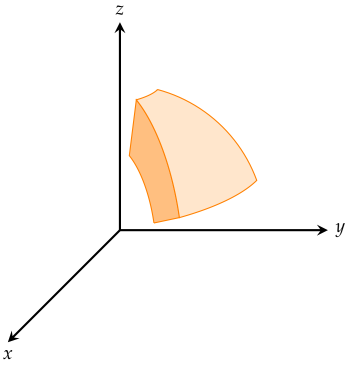


- $a \leq \rho \leq b$ means the shape lies between spheres of radius a and b
- $\alpha \leq \theta \leq \beta$ restricts the shape to a wedge-shaped region over the xy plane
- $c \leq \phi \leq d$ restricts the shape to the space between two cones about the z-axis

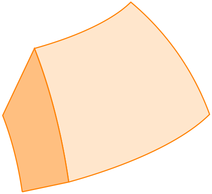
Describing Regions in Spherical Coordinates

Can you sketch each of these regions?

1. $0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi/6, \quad 0 \leq \theta \leq \pi$
2. $1 \leq \rho \leq 2, \quad \pi/2 \leq \phi \leq \pi$
3. $2 \leq \rho \leq 4, \quad 0 \leq \phi \leq \pi/3, \quad 0 \leq \theta \leq \pi$

Triple Integrals in Spherical Coordinates

We need to find the volume of a small spherical wedge



Volume comes from

$$dV =$$

Triple Integrals in Spherical Coordinates

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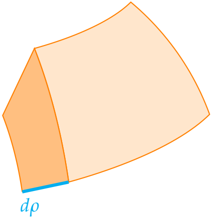

Volume comes from

- Change in ρ

$$dV = d\rho$$

Triple Integrals in Spherical Coordinates

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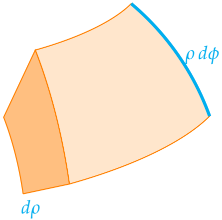

Volume comes from

- Change in ρ
- Change in ϕ

$$dV = \rho\, d\rho\, d\phi$$

Triple Integrals in Spherical Coordinates

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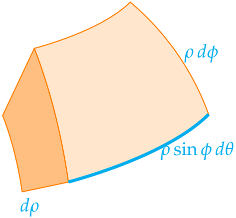

Volume comes from

- Change in ρ
- Change in ϕ
- Change in θ

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Triple Integrals in Spherical Coordinates

$$\iint_E f(x,y,z)\,dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho cos\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

if E is a spherical wedge

$$E = \{(\rho,\theta,\phi) : a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

1. Find $\iiint_E y^2 z^2\,dV$ if E is the region above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$
2. Find $\iiint_E y^2 dV$ if E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $y \geq 0$
3. Find $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ if E lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $\rho = 1$ and $\rho = 2$

Halloween Homework

Determine the volume of pumpkin rind scooped out to form the wicked grin on the pumpkin shown in the figure below.

Hint: Set up the triple integral using corrugated spheroidal coordinates.

Happy Halloween!