Math 213 - Vector Fields

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Reminders

- Homework C5 on 15.9 (change of variables) is due tonight
- Quiz #8 on 15.7–15.8 (triple integrals in spherical and cylindrical coordinates) takes place tomorrow
- Homework C6 on 16.1 (vector fields) is due Friday

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

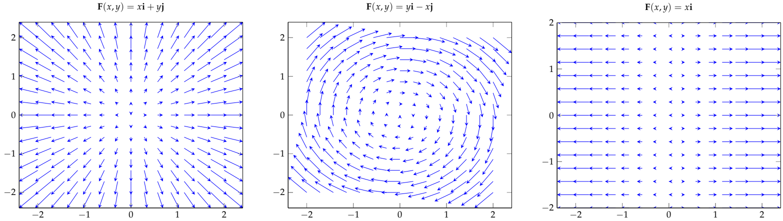
- Understand and Visualize Vector Fields
- Know what the *gradient vector field* of a function is

Vector Fields in the Plane

A **vector** field on $\mathbb{R}^2$ is a function that associates to each (x, y) a *vector*

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

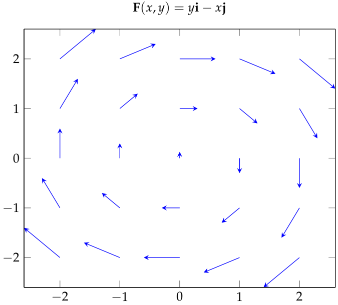
We can visualize a vector field by a *field plot*



Breaking It Down

Where do these plots come from? Consider

$$\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j}$$



$\langle x,y\rangle$	$\mathbf{F}(x,y)$	$\langle x,y\rangle$	$\mathbf{F}(x,y)$
$\langle 1,0\rangle$	$\langle 0,-1\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$
$\langle 1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$	$\langle -1,1\rangle$
$\langle 1,-1\rangle$	$\langle -1,-1\rangle$	$\langle -1,1\rangle$	$\langle 1,1\rangle$
$\langle 2,0\rangle$	$\langle 0,-2\rangle$	$\langle 0,2\rangle$	$\langle 2,0\rangle$
$\langle 2,2\rangle$	$\langle 2,-2\rangle$	$\langle -2,-2\rangle$	$\langle -2,2\rangle$
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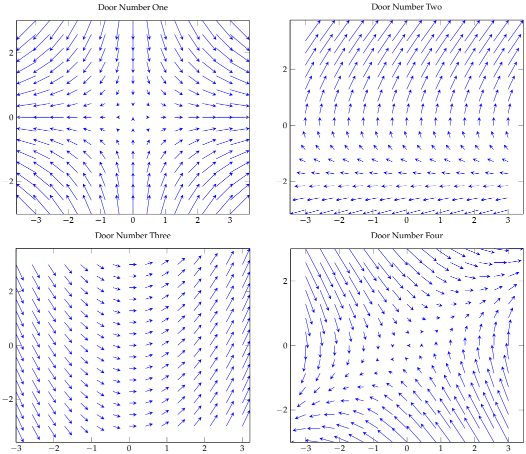
Notice that $\mathbf{F}(x,y)$ is always perpendicular to the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

Mix and Match

Can you match the vector field with its field plot?

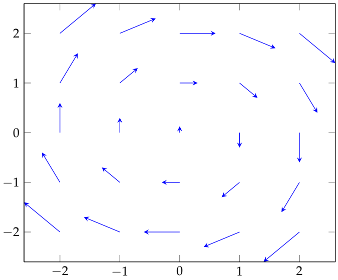
A	$\mathbf{F}(x,y) = \langle x, -y \rangle$	**B**	$\mathbf{F}(x,y) = \langle y, x-y \rangle$
C	$\mathbf{F}(x,y) = \langle y, y+2 \rangle$	**D**	$\mathbf{F}(x,y) = \langle \cos(x+y), x \rangle$



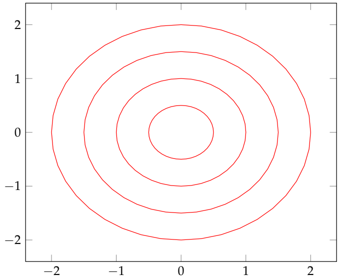
Flow Lines

An important example of a vector field is the *velocity field* of a fluid. The vector $\mathbf{v}(x, y)$ is the velocity vector for the fluid at the point $\langle x, y\rangle$ in the plane. Given the velocity field, you can find the *flow lines* of the fluid – the paths that fluid particles take.

Velocity Field $\mathbf{v}(x, y) = y\mathbf{i} - x\mathbf{j}$



Flow lines of Velocity Field



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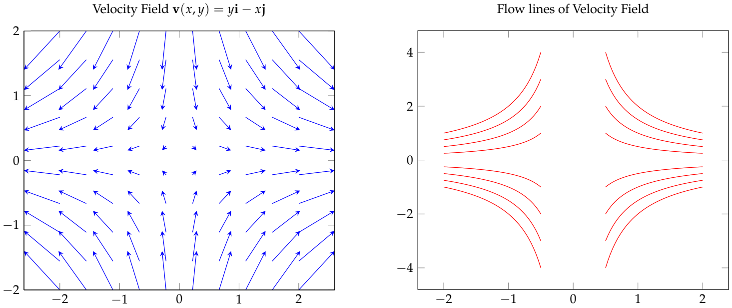
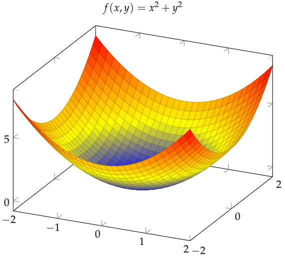


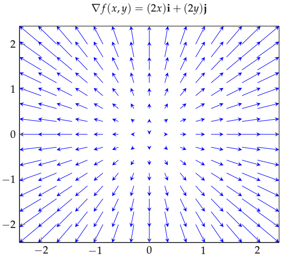
The Gradient Vector Field

If $f(x, y)$ is a function two variables, the *gradient vector field*

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

moves in the direction of greatest change of f





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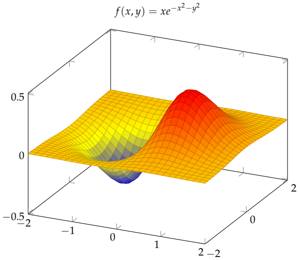
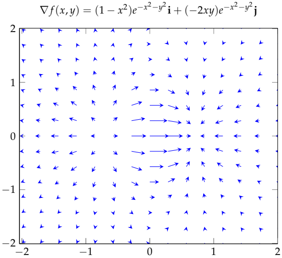




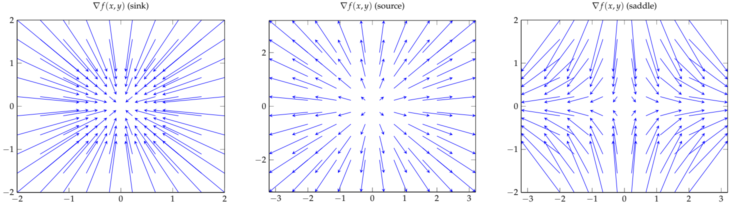
Mix and Match

Can you match these functions with the plots of their gradient vector fields?

A $f(x,y) = x^2 + y^2$ (has a global minimum)

B $f(x,y) = x^2 - y^2$ (has a saddle point)

C $f(x,y) = -(x^2 + y^2)$ (has a global maximum)



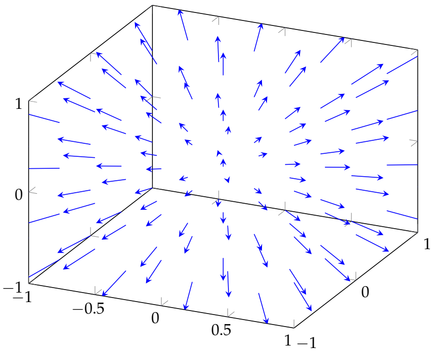
Vector Fields in Space

A **vector field** in space is a function that associates to each (x, y, z) a *vector*

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

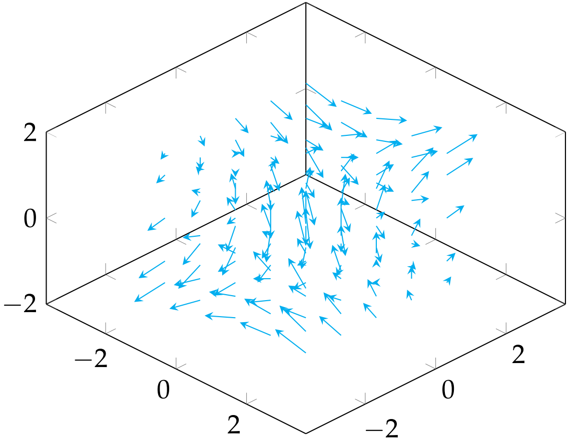
We can visualize a vector field by a *field plot*

$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



Vector Fields in Space

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$



Vector Fields in Physics

1. The electric field generated by a point charge q at the origin is

$$\mathbf{E}(\mathbf{x}) = \frac{q\mathbf{x}}{|\mathbf{x}|^3}$$

2. The gravitational force exerted on a mass m at position $\mathbf{x}$ by a mass M at the origin is

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm\mathbf{x}}{|\mathbf{x}|^3}$$

3. A *conservative force* $\mathbf{F}$ is the gradient of a *potential function* φ, i.e.,

$$\mathbf{F} = \nabla\phi$$

Magnetic Field Lines

Watch A Video