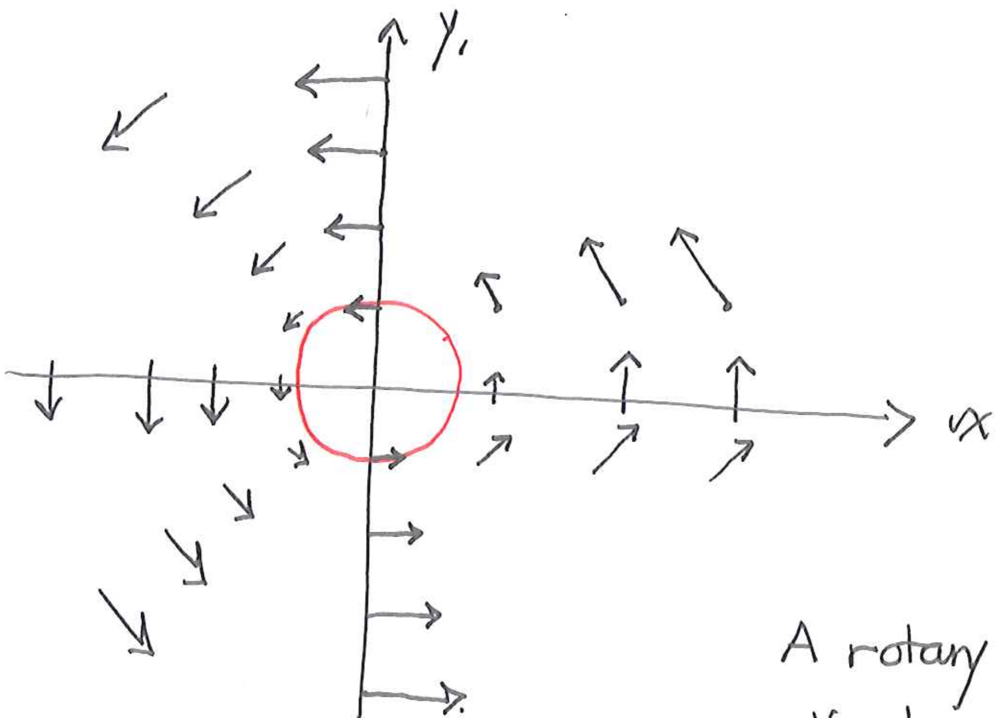
Vector Fields

def. $D \subseteq \mathbb{R}^2$ a plane region

A vector field is a function $\vec{F}$ that assigns to each point $(x,y) \in D$ a vector $\vec{F}(x,y) \in \mathbb{R}^2$

ex. In $\mathbb{R}^2$ define.

$$\vec{F}(x,y) = \left\langle \frac{-1}{4} y \,,\, \frac{1}{4} x \right\rangle.$$



A rotary vector field

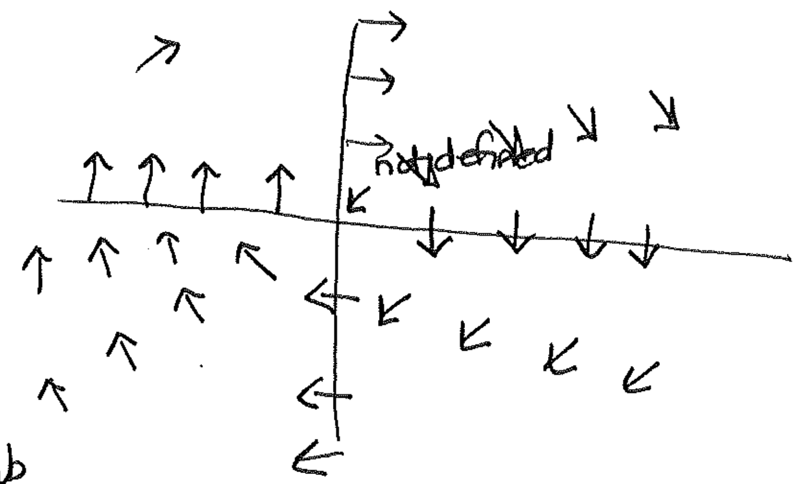
Note: Each is tangent to a circle

Position vector $\vec{x} = \langle x, y \rangle \perp \vec{F} = \langle -\frac{1}{4} y, -\frac{1}{4} x \rangle$

$\vec{x} \cdot \vec{F} = 0.$ $\quad |\vec{F}| = \frac{1}{4}$ radius of circle ex. In $\mathbb{R}^2$

$$\vec{V}(x,y) = \frac{y}{x^2+y^2}\,\vec{i} \quad \frac{-x}{x^2+y^2}\,\vec{j}$$

models planar part of water going down hole in bottom of tub



Note $|\vec{V}| = 1$ at any pt $\neq (0,0)$.

ex. $\vec{F} = \langle y, -x+y \rangle$

(use on-line tool)

www.desmos.com/calculator/eijhparfmd

In $\mathbb{R}^3$

ex. Gravitational force field.

Newton's law of gravity

$$\vec{F} = -\frac{mMG}{r^3}\langle x, y, z\rangle, \qquad r = \sqrt{x^2+y^2+z^2}$$

$$= \left\langle -\frac{mMG}{r^3}x,\ -\frac{mMG}{r^3}y,\ -\frac{mMG}{r^3}z \right\rangle.$$

(Here need $\langle x, y, z\rangle$ with $|\langle x, y, z\rangle| > r_{Earth}$).

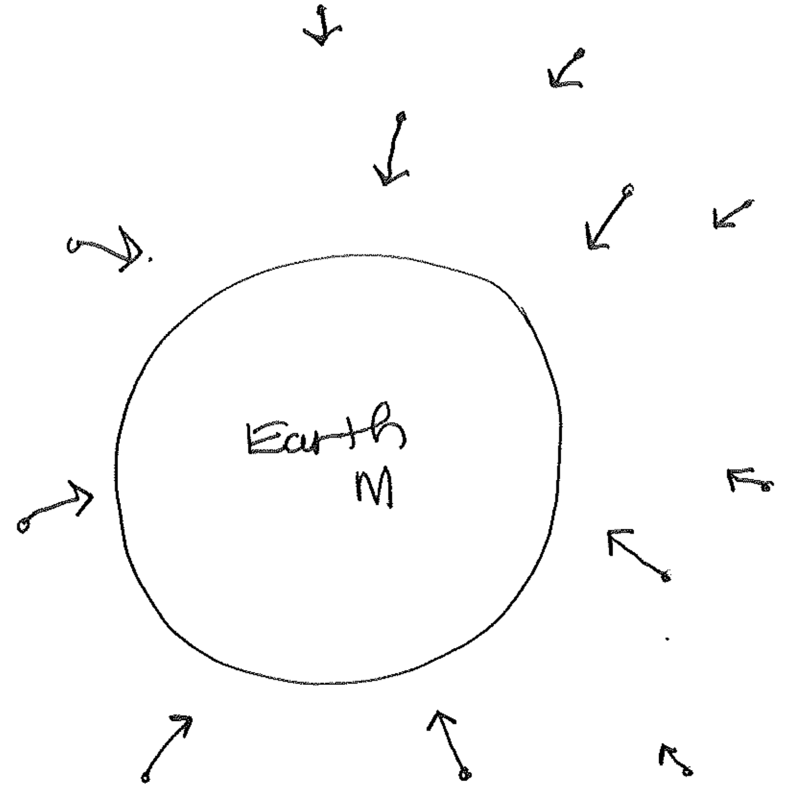
This is a <u>gradient field</u>

$$\vec{F} = \nabla\left(+ \frac{GmM}{\sqrt{x^2+y^2+z^2}} \right).$$

def. A vector field is called a <u>conservative vector field</u> if it is the gradient of some scalar function

$$\vec{F} = \nabla f.$$

f is called the <u>potential function</u>.



ex. Show $\vec{V} = y\vec{i} - x\vec{j}$ is not a gradient vector field.

Sol'n: $\nabla f = \langle f_x, f_y \rangle = \langle y, -x \rangle$

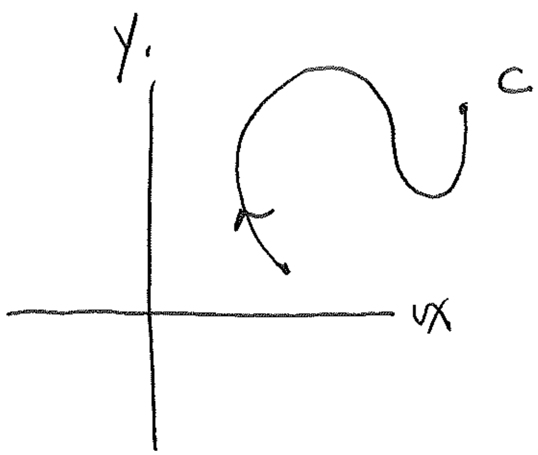
$\Rightarrow f_x = y \qquad f_y = -x.$

f is C^1. (so continuous 1st & 2nd order partials),

$f_{xy} = 1 \quad \& \quad f_{yx} = -1$

impossible.

Line Integrals



plane curve C.

$x = x(t)$

$y = y(t)$

$a \le t \le b$.

(or

$\vec{r}(t) = \langle x(t), y(t) \rangle$

Want to integrate over the smooth curve C.

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Line Integral, 2

ex. $C: [0, 4\pi] \to \mathbb{R}^2$

$t \mapsto \langle \cos t, \sin t \rangle$ circle.

$f(x,y) = x^2 + y^2.$

$$\int_C f \, ds = \int_0^{4\pi} (\cos^2 t + \sin^2 t) \cdot \sqrt{c^2 + s^2} \, dt$$

$$= 4\pi$$

~~f(x,y) =~~

ex. $C: [0, 2\pi] \to \mathbb{R}^2$

$t \mapsto \langle \cos t, \sin t, t \rangle$ helix

$f(x,y,z) = x^2+y^2+z^2$ $\qquad$ $C = (\quad).$

$$\int_C f(x,y,z)\, ds. = \int_a^b f(x(t),y(t),z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt$$

$$\| $$

$$\int_0^{2\pi} (\cos^2 t+\sin^2 t+t^2) \cdot \underbrace{\sqrt{\cancel{dt}(-\sin t)^2 + (\cos^2 t)+1}}_{\sqrt{2}}\, dt$$

$$= \int_0^{2\pi} (1+t^2) \cdot \sqrt{2}\, dt.$$

$$= \sqrt{2} \cdot \left(t+\frac{t^3}{3}\right)\Big]_0^{2\pi}$$

$$= \sqrt{2} \cdot \left(2\pi + \frac{8\pi^3}{3}\right).$$