Math 213 - Line Integrals, Part II

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Reminders

- Exam III is this Wednesday, November 13, 5:00-7:00 PM
- The Review Session for Exam III is tonight, 6:00-8:00 PM in room KAS 213

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

- Know how to compute line integrals of a vector function in the plane and in space

Where Are We?

We've talked about line integrals of *scalar functions* over plane and space curves. Today we'll talk about line integrals of *vector functions* over plane and space curves. If C is a curve in the plane or in space, we've already seen how to compute:

- $\int_C f(x,y)\, ds$, the integral of a scalar function over a plane curve C
- $\int_C f(x,y,z)\, ds$, the integral of a scalar function over a space curve C

Today we'll discuss:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$, the integal of a vector function $\mathbf{F}(x,y)$ over a plane curve C
- $\int_C \mathbf{F} \cdot d\mathbf{r}$, the integral of a vector function $\mathbf{F}(x,y,z)$ over a space curve C

Remember Space Curves?

A *space curve* is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

The *tangent vector* to a space curve is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Recall that $\mathbf{r}'(t)$ is the velocity, and $|\mathbf{r}'(t)|$ is the speed.

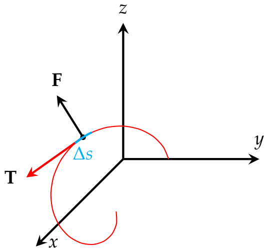
The *unit tangent vector* is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|}.$$

Find the tangent vector and unit tangent vector to the curve

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

for $t = 0$, $t = \pi/2$, and $t = \pi$.



Recall that the work done by a constant force **F** moving an object through a displacement **D** is

$$W = \mathbf{F} \cdot \mathbf{D}$$

What if **F** and the displacement **D** vary as the force acts through a curve C?

Write $\mathbf{D} = \mathbf{T}\Delta s$ where **T** is the tangent vector and Δs is arc length.

Then

$$W \simeq \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}_i \, \Delta s$$

$$\to \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

How Do You Compute It?

The work done by a variable force **F** moving a particle along a curve C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

If C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \leq t \leq b$:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and

$$ds = |\mathbf{r}'(t)|\, dt$$

So

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt \\ &= \int_C \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt\end{aligned}$$

This line integral is sometimes written

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for short

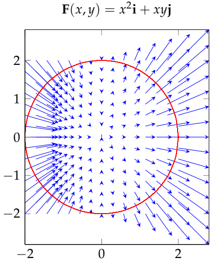
Now You Try It

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ if:

1. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}$, $0 \le t \le 2$
2. $\mathbf{F}(x,y,z) = yze^x\mathbf{i} + zxe^y\mathbf{j} + xye^z\mathbf{k}$ and $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}$, $0 \le t \le \pi/4$

Now You Try It



Find the work done by the force field

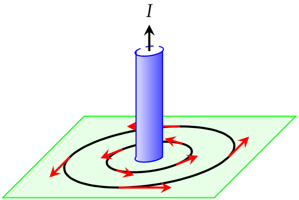
$$\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$$

on a particle that moves around the circle

$$x^2 + y^2 = 4$$

oriented in the counterclockwise direction

Real Science



Steady current in a wire generates a magnetic field $\mathbf{B}$ tangent to any circle that lies in the plane perpendicular to the wire centered on the wire. According to *Ampere's law*,

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where

- I is the net current flowing through the wire
- μ_0 is a physical constant

What is the magnitude of the magnetic field at a distance r from the wire?

Summary

Arc length differential

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

Line integral with respect to arc length

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))ds$$

Line integral with respect to x, y, z

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))y'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt$$

Line integral of a vector field

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot \mathbf{r}'(t)\,dt$$

Chain Rule Puzzler

If $\mathbf{F}(x, y, z)$ is a vector field and $\mathbf{r}(t) = x(t), y(t), z(t))$ is a parameterized curve, what is

$$\frac{d}{dt}\left[F(x(t), y(z), z(t))\right]$$

in terms of ∇F and $\mathbf{r}'(t)$?

Remember the Fundamental Theorem of Calculus?

What is

$$\int_a^b \frac{d}{dt} F(t)\, dt \,?$$

Line Integral of a Gradient Vector Field

Suppose $\mathbf{F} = \nabla\phi$ for a potential function $\phi(x, y, z)$

Suppose $\mathbf{r}(t)$, $a \leq t \leq b$ is a parameterized path C.

Is there a simple way to compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}?$$