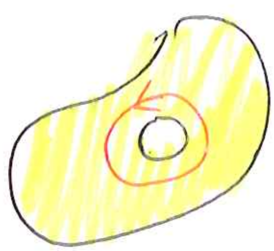
Green's Theorem

11/18/19

Not connected

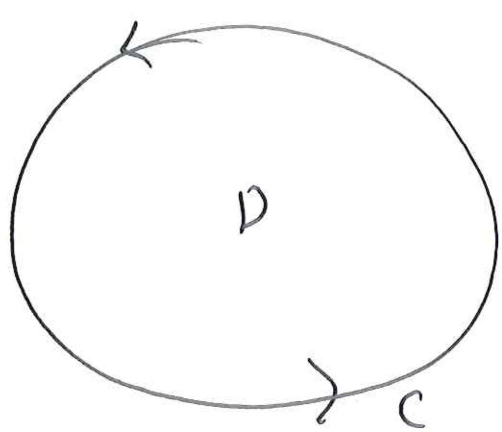
connected

simply connected



Not simply connected

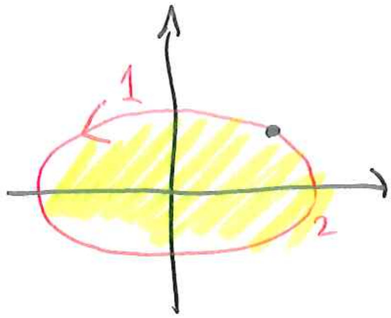
si



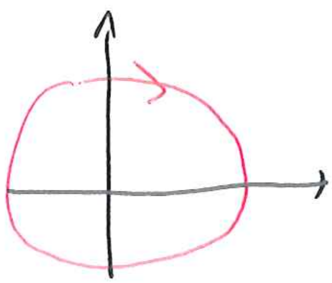
D region

C bounding curve

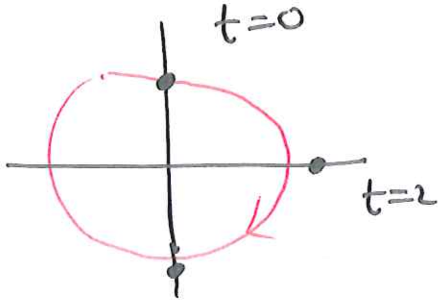
Parameterization



$$x(t) = 2\cos(t)$$
$$y(t) = \sin(t)$$
$$0 \le t \le 2\pi$$

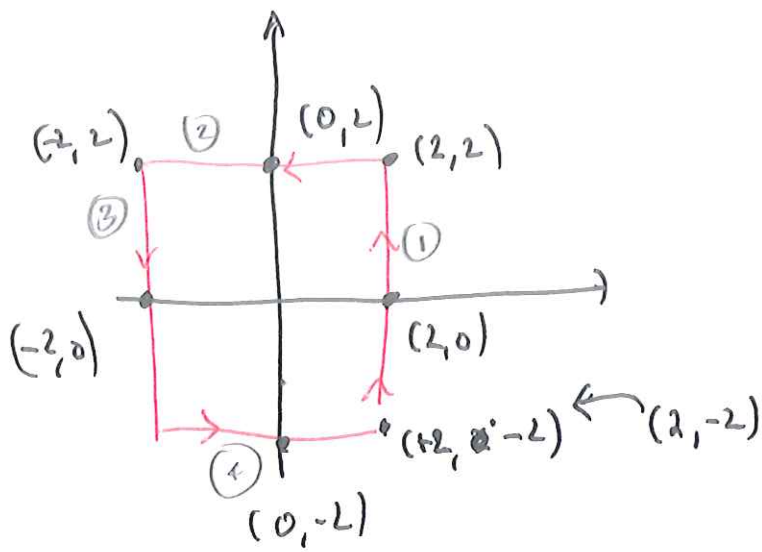


$$x(t) = 2\cos(2\pi - t)$$
$$y(t) = \sin(2\pi - t)$$



$$x(t) = 2\sin(t)$$
$$y(t) = \cos(t)$$

t	$x(t)$	$y(t)$
0	0	1
$\pi/2$	2	0
π	0	-1



1 $\quad x(t) = 2$

$\quad\; y(t) = -2 + 4 \cdot t \qquad 0 \le t \le 1$

2 $\quad x(t) = 2$

$\quad\; y(t) = 2 - 4t \qquad 0 \le t \le 1$

FTC

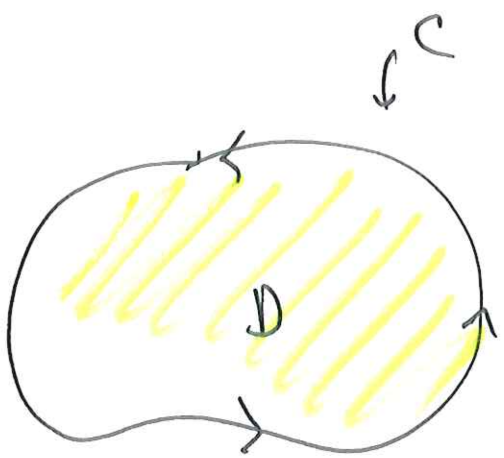
$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Green

$$\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_{(2)}}^{(1)} \vec{F}\cdot d\vec{r}$$

$$= \int_C P\,dx + Q\,dy$$



$$\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

$$\int_C \vec{F} \cdot \vec{dr}$$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

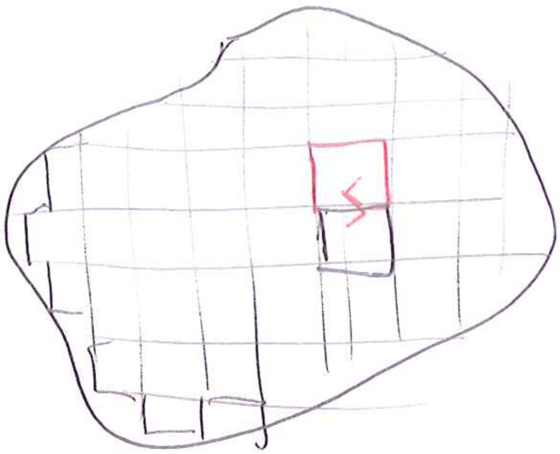
$$a \le t \le b$$

$$= \int_a^b \langle P(x(t), y(t)),\ Q(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

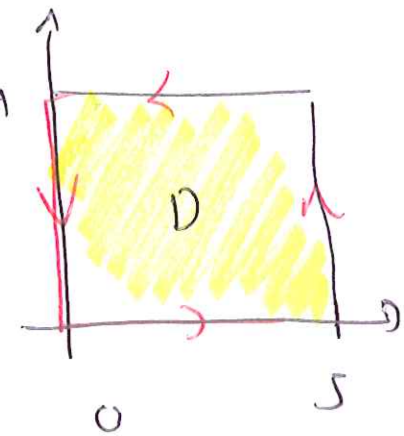
$$= \overset{1}{\int_a^b P(x(t), y(t))\, \underbrace{x'(t)\, dt}_{dx}}$$

$$+ \overset{2}{\int_a^b Q(x(t), y(t))\, \underbrace{y'(t)\, dt}_{dy}}$$

$$= \overset{1}{\int_C P\, dx} + \overset{2}{\int_C Q\, dy}$$



$$\oint_C \left(y^2\,dx + x^2y\,dy\right)$$



$$P(x,y) = y^2$$

$$Q(x,y) = x^2y$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy - 2y$$

$$\oint_C y^2\,dx + x^2y\,dy \overset{\text{Green}}{=} \iint_D \left(2xy - 2y\right)\,dx\,dy$$

$$= \int_0^5 \left(\int_0^4 \left(2xy - 2y\right) dy \right) dx$$

$$= \int_0^5 \left[xy^2 - y^2 \right]\Big|_{y=0}^{y=4}\,dx$$

$$= \int_0^5 [16x - 16] \, dx$$

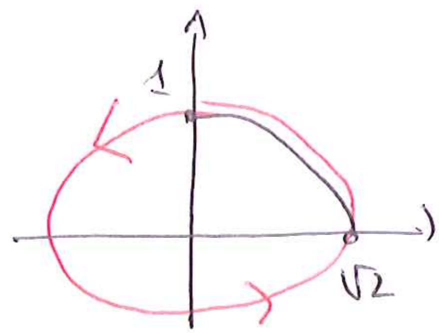
$$= 8x^2 - 16x \Big|_{x=0}^{x=5}$$

$$= 8 \cdot 25 - 16 \cdot 5$$

$$= 200 - 80$$

$$= 120$$

$$\oint_C y^4 \, dx + 2xy^2 \, dy$$



$P(x,y) = y^4$

$Q(x,y) = 2xy^3$

Ellipse
semi-major $\sqrt{2}$
semi-minor 1

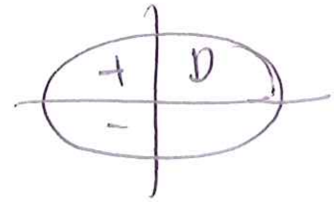
$\cancel{\frac{\partial Q}{\partial y} = 4xy}$

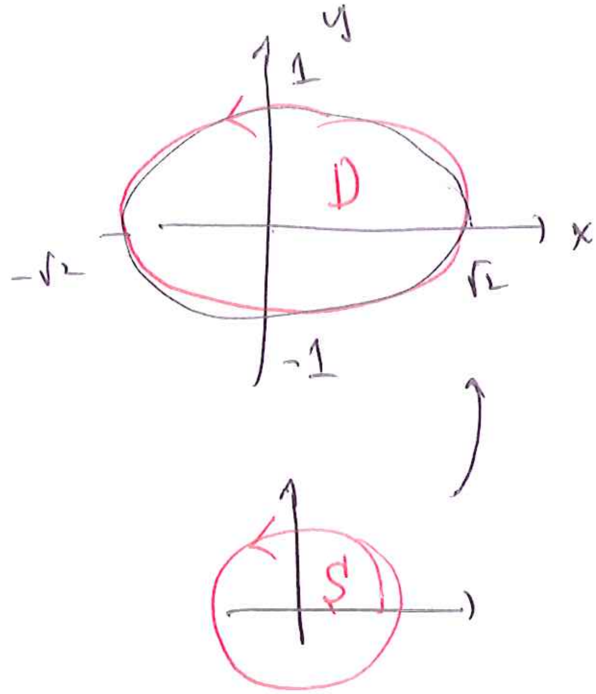
$\cancel{\frac{\partial Q}{\partial x} = 2y^2 \qquad \frac{\partial P}{\partial y} = 4xy^2}$

$$\frac{\partial Q}{\partial x} = 2y^3 \qquad \frac{\partial P}{\partial y} = 4y^3$$

$$\oint_C y^4\,dx + 2xy^2\,dy =$$

$$\iint_D \left(2y^3 - 4y^3\right)\,dx\,dy$$





$x = \sqrt{2}\,u$

$y = 1\,v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{2}$$

$$\iint_D \left(-2y^3\right)\,dx\,dy =$$

c/g of vbls

$$\iint_S \left(-2v^3\right)\sqrt{2}\,du\,dv =$$

polar coords

$$\int_0^{2\pi}\int_0^1 -2(r\sin\theta)^3\,\sqrt{2}\,\underline{r}\,dr\,d\theta =$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

---

1) $\int_D x \, dA$

what Q, P have

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x$?

$\boxed{Q = \frac{1}{2}x^2} \quad \boxed{P = 0}$

$$\int_D x \, dA = \oint_C \frac{1}{2} x^2 \, dy$$

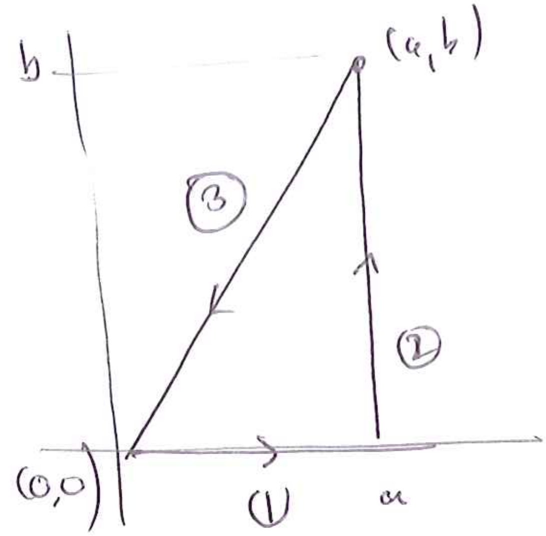
2) $\int_D y \, dA$

what Q, P have

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$

$Q = 0 \quad P = -\frac{y^2}{2}$

$$\int_D y \, dA = \oint_C -\frac{y^2}{2} \, dx$$



$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \qquad A = \frac{1}{2}ab$$

$$\oint_C x^2\,dy = \int_{②} x^2\,dy + \int_{③} x^2\,dy$$

① $x(t) = at \qquad y(t) = 0 \qquad 0 \le t \le 1$

② $x(t) = a \qquad y(t) = tb \qquad 0 \le t \le 1$

③ $\vec{r}(t) = \langle a, b \rangle + t\langle -a, -b \rangle \qquad 0 \le t \le 1$