Curl and Divergence

Hwk D1: The circulation of a vector field $\vec{F}$ around a closed curve C is

$$\oint_C \vec{F} \cdot d\vec{r}$$

$$\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$$

$$\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

$$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ scalar

$$\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$ vector

$$\vec{F} = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial z}\right) +$$

$$\hat{j}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) +$$

$$\hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Note: If $R = 0$, i.e., $\vec{F} = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{\imath} \cdot 0 + \hat{\jmath} \cdot 0 + \hat{k} \cdot 0$$

$$= \vec{0}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$= \hat{\imath}\,(0) + \hat{\jmath}\,(0) + \hat{k}\,(-1-1)$$

$$= -2\hat{k}$$

$$\nabla \times (\nabla f) =$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} =$$

$$\hat{\imath}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) +$$

$$\hat{\jmath}\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) +$$

$$\hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= \vec{0}$$

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\vec{F} = \langle P, Q, R \rangle$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\cancel{\vec{\nabla}}$

1) $\vec{P} = \langle x, y, z \rangle$

$$\vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x, y, z \rangle$$

$$= \frac{\partial x}{\partial \cancel{y}x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 3$$

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2) $\vec{P} = \langle y, -x, 0 \rangle$

$$\vec{\nabla} \cdot \vec{P} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle y, -x, 0 \rangle$$

$$= \frac{\partial y}{\partial x} + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(0)$$

$$= 0$$

$$F = P\hat{i} + Q\hat{j} + R\hat{k}$$

div curl F =

$$\text{div}\left[\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) +$$

$$\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) +$$

$$\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= 0$$

$$\vec{F}(x,y,z) = y^2z^3\,\hat{i} + 2xyz^3\,\hat{j} + 3xy^2z^2\,\hat{k}$$

find f so $\nabla f = \vec{F}$

— $\dfrac{\partial f}{\partial x} = y^2z^3$

$\therefore\ f(x,y,z) = xy^2z^3 + C(y,z)$

— $\dfrac{\partial f}{\partial y} = 2xyz^3$ —(1)

$\dfrac{\partial f}{\partial y} = 2xyz^3 + \dfrac{\partial C}{\partial y}$ —(2)

By (1), (2), $\dfrac{\partial C}{\partial y} = 0$

— $\dfrac{\partial f}{\partial z} = 3xy^2z^2$

$\dfrac{\partial f}{\partial z} = 3xy^2z^2 + \dfrac{\partial C}{\partial z}$

$\dfrac{\partial C}{\partial z} = 0$ $\boxed{f(x,y,z) = \cancel{4}\,xy^2z^3 + C}$

$$\operatorname{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f\,\vec{\nabla}\cdot\vec{F}$$

check:

$$\vec{F} = \langle P, Q, R\rangle$$

$$f\vec{F} = \langle fP, fQ, fR\rangle$$

$$\vec{\nabla}\cdot(f\vec{F}) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

$$\begin{aligned} = \;& \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} \; + \\ & \frac{\partial f}{\partial y}Q + f\frac{\partial Q}{\partial y} \; + \\ & \frac{\partial f}{\partial z}R + f\frac{\partial R}{\partial z} \end{aligned}$$

$$= \nabla f\cdot\vec{F} + f\cdot(\nabla\cdot\vec{F})$$

curl $(f\vec{F}) =$

$$\vec{\nabla} f \times \vec{F} + f\vec{\nabla} \times \vec{F}$$

div (grad f) =

$$\operatorname{div} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =$$

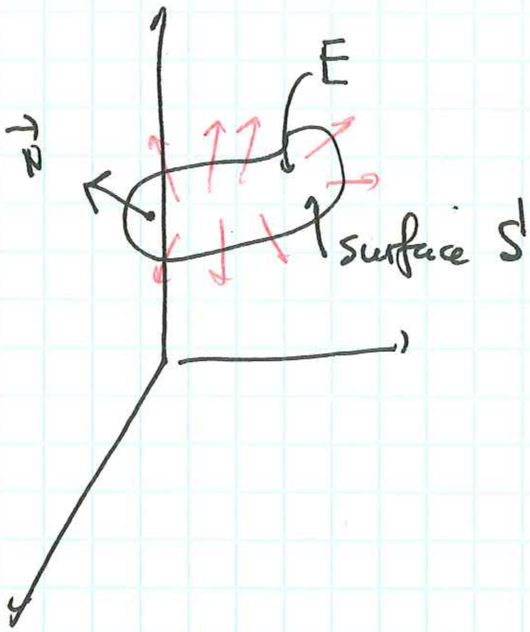
$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

Laplacian of f

Important for heat and wave equations
in classical physics

Preview

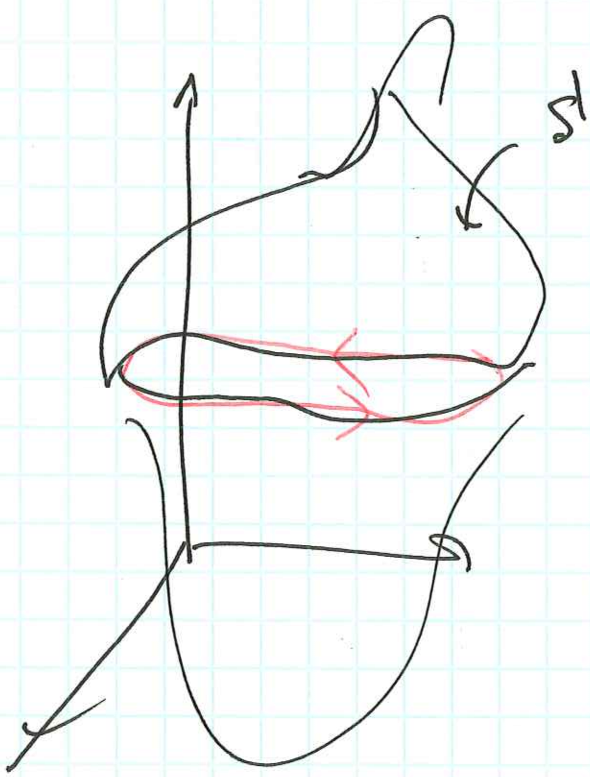


Divergence Theorem

$$\cancel{\iiint_E \vec{F}\cdot\vec{N}\, dV}$$

$$\iint_S \vec{F}\cdot\vec{N}\, dS =$$

$$\iiint_E (\operatorname{div} \vec{F})\, dV$$



Stokes' Theorem

$$\int_C \vec{F}\cdot d\vec{r} =$$

$$\iint_S \vec{\nabla}\times\vec{F}\cdot N\, dS$$