Math 213 - Parametric Surfaces and their Areas

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November 22, 2019

Reminders

- Homework D1 is due on Friday of this week
- Homework D2 is due on **Monday** of next week
- Thanksgiving is coming!

Unit IV: Vector Calculus

Goals of the Day

This lecture is about parametric surfaces. You'll learn:

- How to define and visualize parametric surfaces
- How to find the tangent plane to a parametric surface at a point
- How to compute the surface area of a parametric surface using double integrals

Parametric Curves and Parametric Surfaces

Parametric Curve

A parametric curve in $\mathbb{R}^3$ is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $a \leq t \leq b$

There is *one parameter*, because a curve is a *one-dimensional* object

There are *three component functions*, because the curve lives in *three-dimensional* space.

Parametric Surface

A parametric surface in $\mathbb{R}^3$ is given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where (u,v) lie in a region D of the uv plane.

There are *two parameters*, because a surface is a *two-dimensional* object

There are *three component functions*, because the surface lives in *three-dimensional* space.

You Are Living on a Parametric Surface

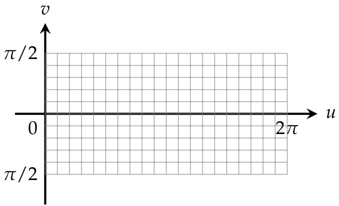
Let u be your longitude (in radians, for this course)

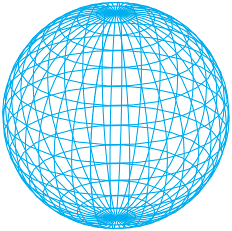
Let v be your latitude (in radians)

Let R be the radius of the Earth

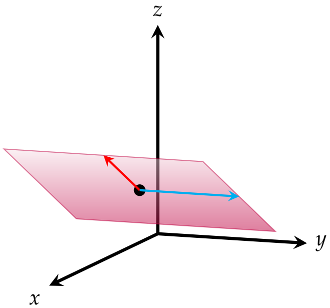
Your position is

$$\mathbf{r}(u,v) = R\cos(v)\cos(u)\mathbf{i} + R\cos(v)\sin(u)\mathbf{j} + R\sin(v)\mathbf{k}$$



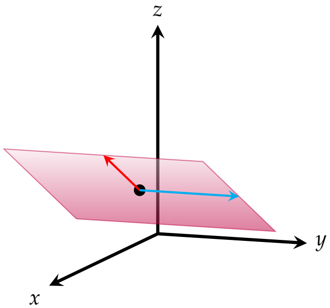


More Parameterized Surfaces: Planes



Problem: Find a parametric representation for the plane through $\langle 1, 0, 1\rangle$ that contains the vectors $\langle 2, 0, 1\rangle$ and $\langle 0, 2, 0\rangle$

More Parameterized Surfaces: Planes



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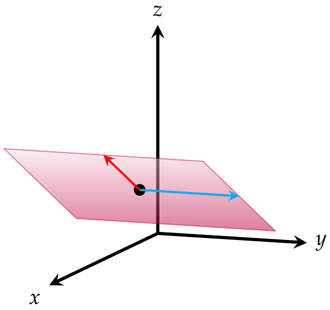
Solution: Let $\mathbf{r}_0 = \langle 1, 0, 1 \rangle$. Any point in the plane is given by

$$\mathbf{r}(s, t) = \langle 1, 0, 1 \rangle + s\langle 2, 0, 1 \rangle + t\langle 0, 2, 0 \rangle$$

More Parameterized Surfaces: Planes



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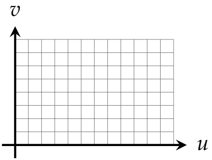
Now you try it:

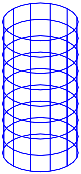
Find a parametric representation for the plane through the point $(0, -1, 5)$ that contains the vectors $\langle 2, 1, 4\rangle$ and $\langle -3, 2, 5\rangle$.

More Parameterized Surfaces: The Cylinder

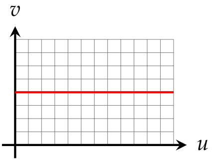


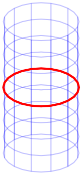
$$\mathbf{r}(u,v) = r\cos(u)\mathbf{i} + r\sin(u)\mathbf{j} + v\mathbf{k}$$
$$D = \{(u,v) : 0 \le u \le 2\pi,\ 0 \le v \le h\}$$

parameterizes a cylinder of radius r and height h



More Parameterized Surfaces: The Cylinder





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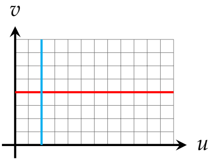
parameterizes a cylinder of radius r and height h

If we fix v and vary u over the cylinder, we trace out a circle

More Parameterized Surfaces: The Cylinder



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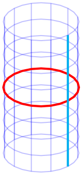
$$D = \{(u,v) : 0 \le u \le 2\pi,\ 0 \le v \le h\}$$

parameterizes a cylinder of radius r and height h

If we fix v and vary u over the cylinder, we trace out a circle

If we fix u and vary v, we trace out a vertical line



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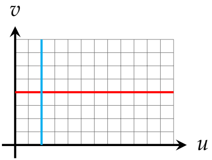

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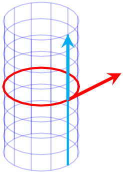


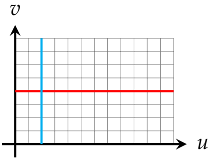
If we fix v and vary u over the cylinder, we trace out a circle

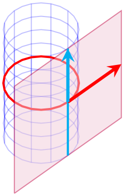
If we fix u and vary v, we trace out a vertical line

Each of these curves has a *tangent vector*:

$$\mathbf{r}_u(u,v) = -r\sin(u)\mathbf{i} + r\cos(u)\mathbf{j}$$
$$\mathbf{r}_v(u,v) = \mathbf{k}$$

More Parameterized Surfaces: The Cylinder





$$\mathbf{r}(u,v) = r\cos(u)\mathbf{i} + r\sin(u)\mathbf{j} + v\mathbf{k}$$
$$D = \{(u,v) : 0 \le u \le 2\pi,\ 0 \le v \le h\}$$

parameterizes a cylinder of radius r and height h

The two tangent vectors

$$\mathbf{r}_u(u,v) = -r\sin(u)\mathbf{i} + r\cos(u)\mathbf{j}$$
$$\mathbf{r}_v(u,v) = \mathbf{k}$$

span the *tangent plane* to the cylinder at the given point

The Tangent Vectors $\mathbf{r}_u$ and $\mathbf{r}_v$

Suppose

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D$$

is a parameterized surface.

At a point $P_0 = \mathbf{r}(u_0, v_0)$, the vectors

$$\mathbf{r}_u(u_0,v_0) = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}$$

$$\mathbf{r}_v(u_0,v_0) = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}$$

are *both* tangent to the surface.

The Tangent Plane

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

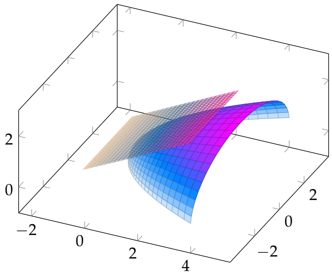
The *tangent plane* to a parameterized surface at $P_0 = \mathbf{r}(u_0, v_0)$ is the plane passing through P_0 and perpendicular to $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

Find the equation of the tangent plane to the surface

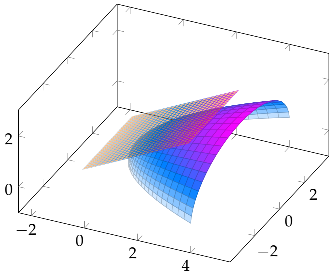
$$\mathbf{r}(u, v) = u^2\mathbf{i} + 2u \sin v\mathbf{j} + u \cos v\mathbf{k}$$

at $u = 1, v = 0$.

The Tangent Plane

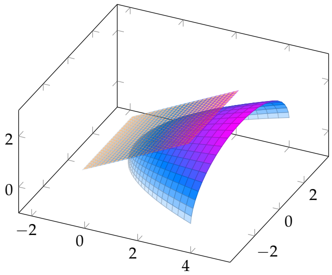


The Tangent Plane



$$\mathbf{r}(u,v) = \langle u^2, 2u\sin v, u\cos v\rangle$$
$$\mathbf{r}_u(u,v) = \langle 2u, 2\sin v, \cos v\rangle$$
$$\mathbf{r}_v(u,v) = \langle 0, 2u\cos v, -u\sin v\rangle$$

The Tangent Plane



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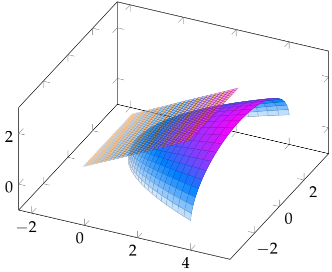
$$\mathbf{r}(1,0) = \langle 1,0,1\rangle$$
$$\mathbf{r}_u(1,0) = \langle 2,0,1\rangle$$
$$\mathbf{r}_v(1,0) = \langle 0,2,0\rangle$$

The Tangent Plane



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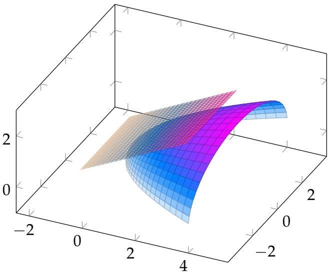
The normal to the plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1,0,2\rangle$$

The Tangent Plane



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The normal to the plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, 0, 2\rangle$$

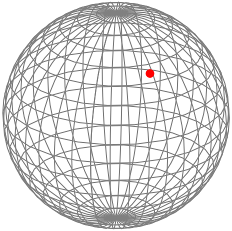
so the equation of the plane is

$$(-1)(x-1) + 2(z-1) = 0$$

The tangent plane to the surface at $(1,0,1)$ is parameterized by

$$\langle 1+2s, 2t, 1+s\rangle$$

The Sphere Revisited



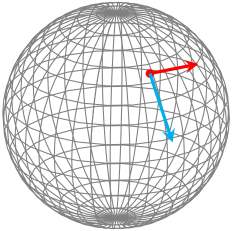
$$\mathbf{r}(u,v) = \sin(v)\cos(u)\mathbf{i} + \sin(v)\sin(u)\mathbf{j} + \cos(v)\mathbf{k}$$

$$0 \le u \le 2\pi,\ 0 \le v \le \pi$$

$$\mathbf{r}_u = -\sin(v)\sin(u)\mathbf{i} + \sin(v)\cos(u)\mathbf{j}$$
$$\mathbf{r}_v = \cos(v)\cos(u)\mathbf{i} + \cos(v)\sin(u)\mathbf{j} - \sin(v)\mathbf{k}$$

Find the tangent plane to the sphere at $(u,v) = (\pi/4, \pi/4)$

The Sphere Revisited



$$\begin{aligned}\mathbf{r}(u,v) &= \sin(v)\cos(u)\mathbf{i} \\ &\quad + \sin(v)\sin(u)\mathbf{j} \\ &\quad + \cos(v)\mathbf{k}\end{aligned}$$

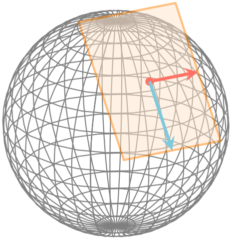
$$0 \le u \le 2\pi,\ 0 \le v \le \pi$$

$$\begin{aligned}\mathbf{r}_u &= -\sin(v)\sin(u)\mathbf{i} + \sin(v)\cos(u)\mathbf{j} \\ \mathbf{r}_v &= \quad \cos(v)\cos(u)\mathbf{i} + \cos(v)\sin(u)\mathbf{j} \\ &\quad -\sin(v)\mathbf{k}\end{aligned}$$

Find the tangent plane to the sphere at $(u,v) = (\pi/4, \pi/4)$

$$\begin{aligned}\mathbf{r}(\pi/4,\pi/4) &= \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k} \\ \mathbf{r}_u(\pi/4,\pi/4) &= -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \\ \mathbf{r}_v(\pi/4,\pi/4) &= \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}\end{aligned}$$

The Sphere Revisited



$$\mathbf{r}(u,v) = \sin(v)\cos(u)\mathbf{i} + \sin(v)\sin(u)\mathbf{j} + \cos(v)\mathbf{k}$$

$$0 \le u \le 2\pi,\ 0 \le v \le \pi$$

$$\mathbf{r}_u = -\sin(v)\sin(u)\mathbf{i} + \sin(v)\cos(u)\mathbf{j}$$
$$\mathbf{r}_v = \cos(v)\cos(u)\mathbf{i} + \cos(v)\sin(u)\mathbf{j} - \sin(v)\mathbf{k}$$

Find the tangent plane to the sphere at $(u,v) = (\pi/4, \pi/4)$

$$\mathbf{r}(\pi/4,\pi/4) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$
$$\mathbf{r}_u(\pi/4,\pi/4) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$
$$\mathbf{r}_v(\pi/4,\pi/4) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$$

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}\right)$$
$$0 = \frac{1}{\sqrt{2}}(x - \frac{1}{2}) + \frac{1}{\sqrt{2}}(y - \frac{1}{2}) + (z - \frac{\sqrt{2}}{2})$$

Sneak Preview

Parametric Curves - Arc Length

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

$$\left|\mathbf{r}'(t)\right| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

$$ds = \left|\mathbf{r}'(t)\right| \, dt$$

$$L = \int_a^b \left|\mathbf{r}'(t)\right| \, dt$$

Parametric Surfaces - Area

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

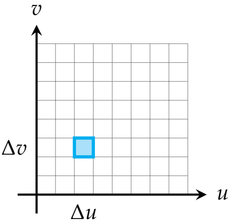
$$\mathbf{r}_u(u,v) = \frac{\partial \mathbf{r}}{\partial u}(u,v)$$

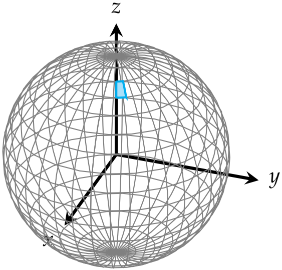
$$\mathbf{r}_v(u,v) = \frac{\partial \mathbf{r}}{\partial v}(u,v)$$

$$dA = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \, du \, dv$$

$$S = \iint_D \left|\mathbf{r}_u \times \mathbf{r}_v\right| \, du \, dv$$

Surface Area





Find the area ΔA of a small patch of surface

The map $(u, v) \mapsto \mathbf{r}(u, v)$ takes the square to a parallelogram with sides $\mathbf{r}_u\,\Delta u$ and $\mathbf{r}_v\,\Delta v$

The area of the parallelogram is

$$|\mathbf{r}_u\,\Delta u \times \mathbf{r}_v\,\Delta v| = |\mathbf{r}_u \times \mathbf{r}_v|\;\Delta u\,\Delta v$$

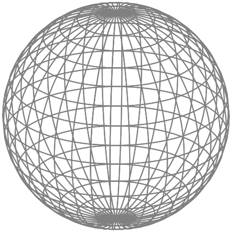
The area of the surface is approximately

$$A = \sum_{i,j} |\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)|\;\Delta u\,\Delta v$$

and exactly

$$\iint_D |\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)|\;du\,dv$$

Surface Area of a Sphere



$$\begin{aligned}\mathbf{r}(u,v) = {} & a\sin(v)\cos(u)\mathbf{i} \\ & + a\sin(v)\sin(u)\mathbf{j} \\ & + a\cos(v)\mathbf{k}\end{aligned}$$

$$0 \le u \le 2\pi, \ 0 \le v \le \pi$$

$$\begin{aligned}\mathbf{r}_u &= -a\sin(v)\sin(u)\mathbf{i} + a\sin(v)\cos(u)\mathbf{j} \\ \mathbf{r}_v &= \quad a\cos(v)\cos(u)\mathbf{i} + a\cos(v)\sin(u)\mathbf{j} \\ & \quad - \sin(v)\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= a^2\sin^2(v)\cos(u)\mathbf{i} + a^2\sin^2(v)\sin(u)\mathbf{j} \ - a^2\cos(v)\sin(v)\mathbf{k} \\ |\mathbf{r}_u \times \mathbf{r}_v| &= a^2\sin^2(v)\end{aligned}$$

Hence

$$S = \int_0^\pi \int_0^{2\pi} a^2 \sin^2 v \, du \, dv = 4\pi a^2$$

Surfaces Area of a Graph

The graph of a function $z = f(x,y)$ is also a parameterized surface:

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

$$\mathbf{r}_x(x,y) = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}$$

$$\mathbf{r}_y(x,y) = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial f}{\partial x}\mathbf{i} + -\frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Hence, the surface area of the graph over a domain D in the xy plane is

$$S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Surface Area of a Graph

The surface area of the graph over a domain D in the xy plane is

$$S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Find the area of the graph of $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$

Curves and Surfaces

Curves

Parameterization

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Tangent

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Tangent line at $t = a$

$$\mathbf{L}(s) = \mathbf{r}(a) + s\mathbf{r}'(a)$$

Arc length differential

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Surfaces

Parameterization

$$\begin{aligned}\mathbf{r}(u,v) = {} & x(u,v)\mathbf{i} + y(u,v)\mathbf{j} \\ & + z(u,v)\mathbf{k}\end{aligned}$$

Tangents

$$\mathbf{r}_u(u,v) = \frac{\partial}{\partial u}\mathbf{r}(u,v)$$
$$\mathbf{r}_v(u,v) = \frac{\partial}{\partial v}\mathbf{r}(u,v)$$

Normal

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$$

Area Differential

$$dA = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$