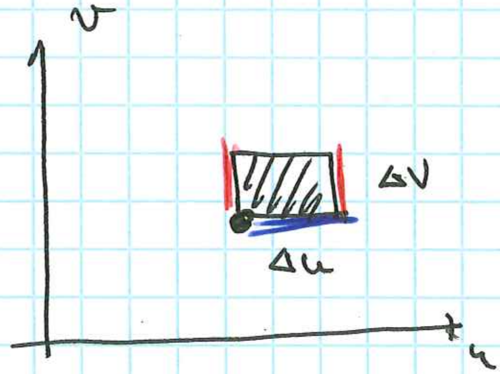
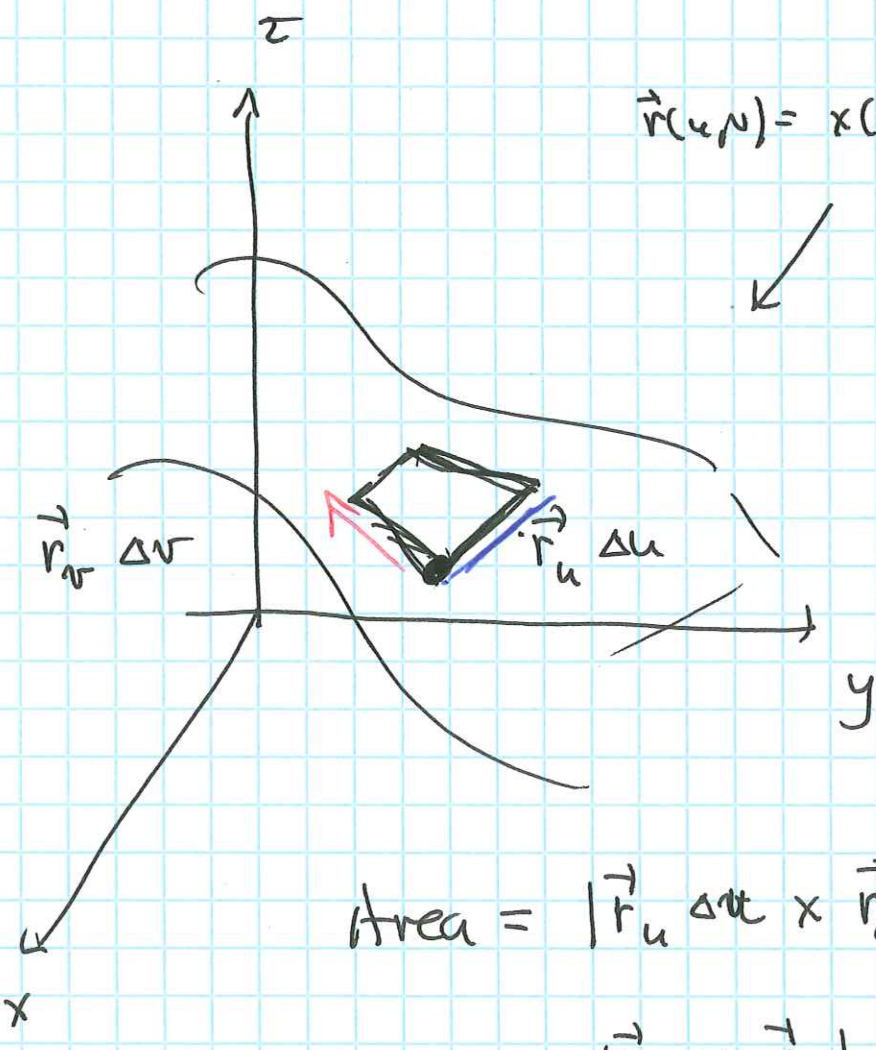
Parameterized Surfaces

11/25/19 ①



$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$



$$\text{Area} = |\vec{r}_u\,\Delta u \times \vec{r}_v\,\Delta v|$$

$$= |\vec{r}_u \times \vec{r}_v|\;du\,dv$$

$$dA = |\vec{r}_u \times \vec{r}_v|\;du\,dv$$

11/25/99 ②

Sphere of radius a

$\vec{r}(u,v) =$

$(a \sin v \cos u)\hat{i} +$

$(a \sin v \sin u)\hat{j} +$

$(a \cos v)\hat{k}$

$u \leftrightarrow \cancel{\phi}$

$v \leftrightarrow \cancel{\phi}$

$u \Leftrightarrow \theta$

$v \Leftrightarrow \varphi$

$a \leftrightarrow \rho$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \langle a \sin v(-\overset{\sin}{\cancel{\cos}}\, u),\ a \sin v(\cos u),\ 0 \rangle$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \langle a \cos v \cos u,\ a \cos v \sin u,\ -a \sin v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin v\, \overset{\sin u}{\cancel{\cos u}} & a \sin v \cos u & 0 \\ a \cos v \cos u & a \cos v \sin u & -a \sin v \end{vmatrix}$$

$$= \langle -a^2 \sin^2 v \cos u,\ -a^2 \sin^2 v \sin u,$$

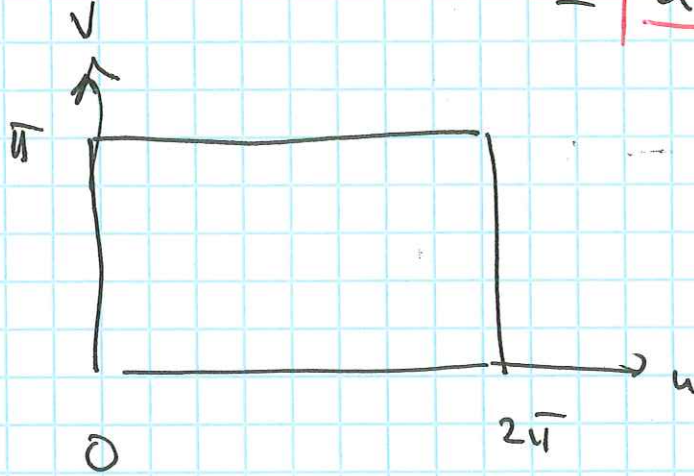
$$-a^2 \sin v \cos v\, \underline{\sin^2 u} + -a^2\, \underline{\cos^2 u} \sin v \cos v \rangle$$

$$= \langle -a^2 \sin^2 v \cos u,\ -a^2 \sin^2 v \sin u,\ -a^2 \sin v \cos v \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\left\{ a^4 \sin^4 v \cos^2 u + a^4 \sin^4 v \sin^2 u + a^4 \cos^2 v \sin^2 v \right\}}$$

$$= a^2 \sin^2 v \times \sqrt{\left\{ \sin^2 v \underline{\cos^2 u} + \sin^2 v \underline{\sin^2 u} + \cos^2 v \right\}}$$

$$= \boxed{a^2 \sin v}$$

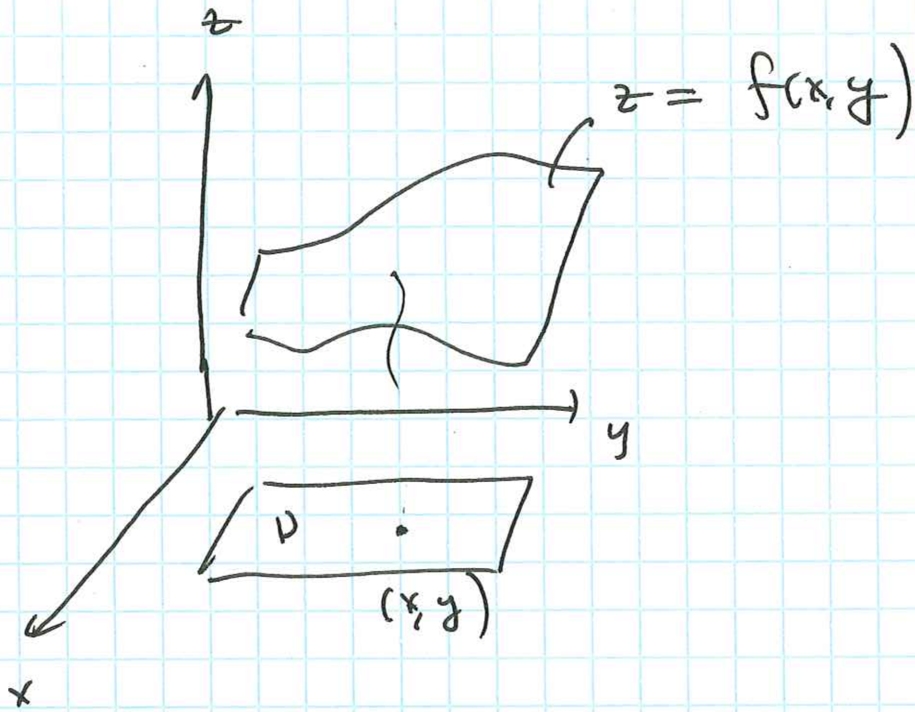


$$S = \int_0^{2\pi} \int_0^{\pi} a^2 \sin v \, dv \, du$$

$$= 2\pi \int_0^{\pi} a^2 \sin v \, dv$$

$$= 2\pi a^2 \left[-\cos v \right]_{v=0}^{v=\pi}$$

$$= 4\pi a^2$$



$$\vec{r}(x,y) = x\,\hat{i} + y\,\hat{j} + f(x,y)\,\hat{k}$$

$$\frac{\partial \vec{r}}{\partial x} = \hat{i} + 0\hat{j} + \frac{\partial f}{\partial x}(x,y)\,\hat{k}$$

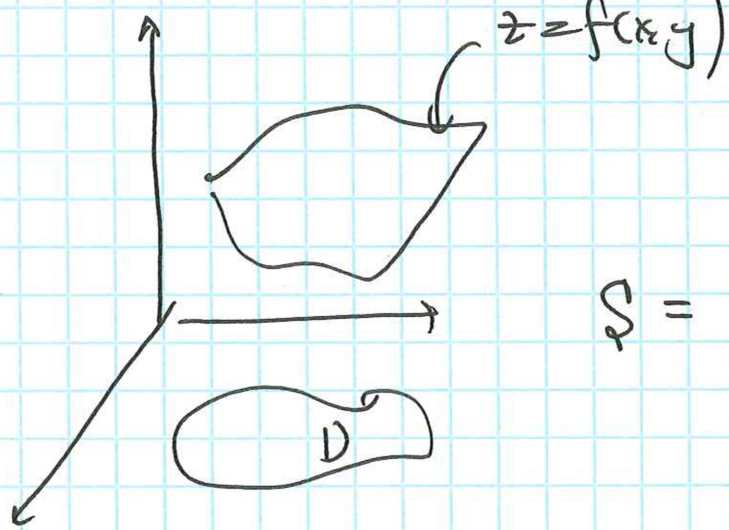
$$\frac{\partial \vec{r}}{\partial y} = 0\hat{i} + \hat{j} + \frac{\partial f}{\partial y}(x,y)\,\hat{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix}$$

$$-\frac{\partial f}{\partial x}\hat{i} + -\frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Area of a Surface

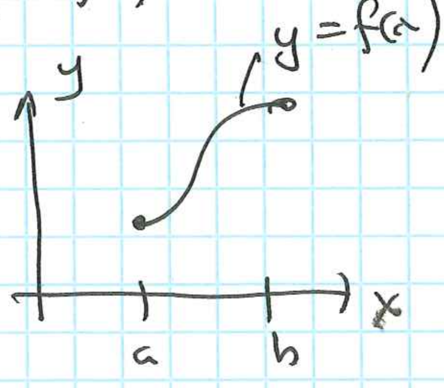


$$S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

Length of a Curve

$y = f(x)$

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$



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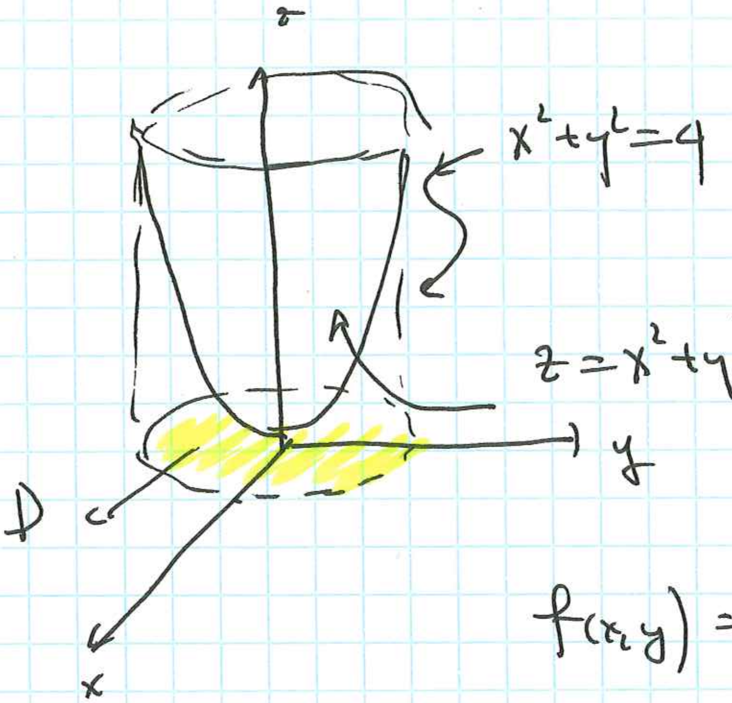


$$f(x,y) = x^2+y^2$$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

D in polar $\quad 0 \le r \le 2$

$$0 \le \theta \le 2\pi$$

$$r^2 = x^2+y^2$$

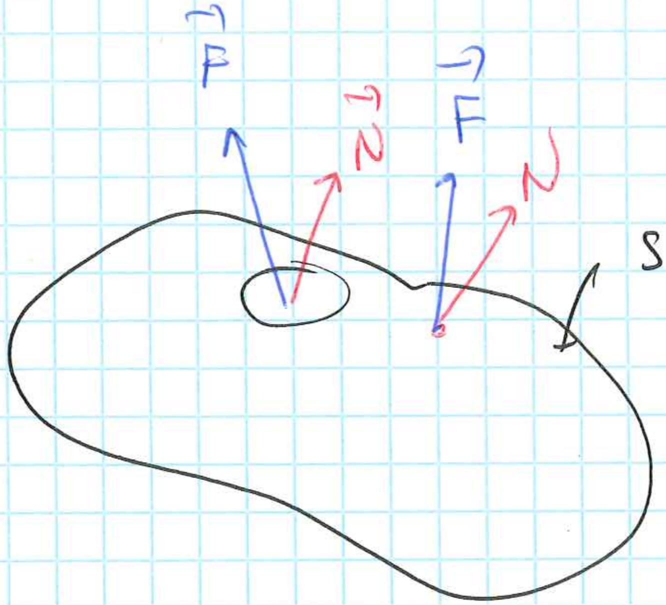
$$S = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$= \int_0^{2\pi} \left(\int_0^2 \sqrt{1+4r^2}\, r\, dr \right) d\theta$$

$$u = 1+4r^2$$
$$du = 8r\, dr$$

$$= 2\pi \left[\frac{1}{8} \int_1^{17} \sqrt{u}\, du \right]$$

$$= \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=17}$$



outward flux $= \vec{F} \cdot \vec{N} \; dA$

Surface Integrals

#1: $f(x,y,z) = g(\sqrt{x^2+y^2+z^2}) \quad g(2) = 5$

$\iint_S f(x,y,z)\, dS$ if $S: x^2+y^2+z^2 = 4$

$\vec{r}(u,v) = \langle 2\sin v\cos u,\ 2\sin v \sin u,\ 2\cos v \rangle$

$f(x(u,v), y(u,v), z(u,v)) = g(2) = 5$

$\iint_S \cancel{\overset{5}{\cancel{g}}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv = 5 \cdot 2} \iint_S |\vec{r}_u \times \vec{r}_v|\, du\, dv$

$\cancel{= 5 \cdot 2 \cdot (4\pi \cdot 2^2)} = \boxed{32\pi}$

$$\iint f(x,y,z)\,dS =$$

$$\iint f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,\cancel{dS}\cdot du\,dv =$$

$$5\iint |\vec{r}_u \times \vec{r}_v|\,du\,dv =$$

$$5\cdot\ 4\pi(2)^2 =$$

$$20\pi\cdot 4 = 80\pi$$

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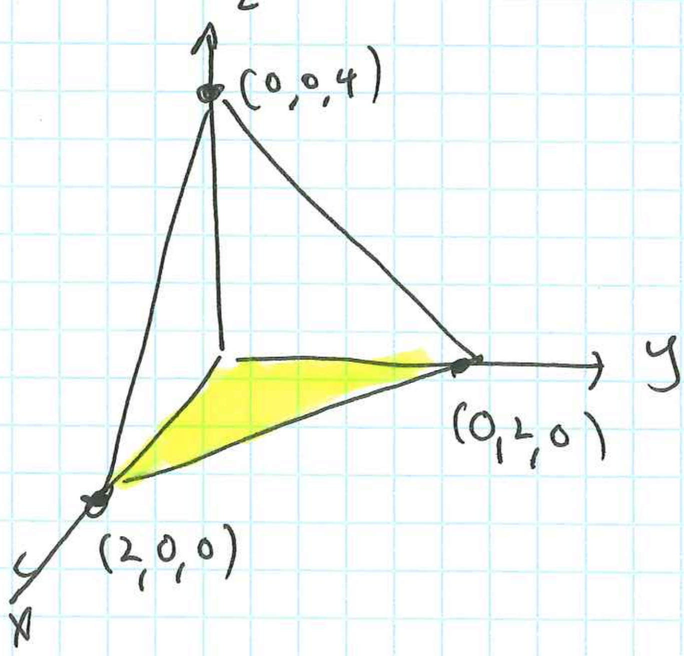
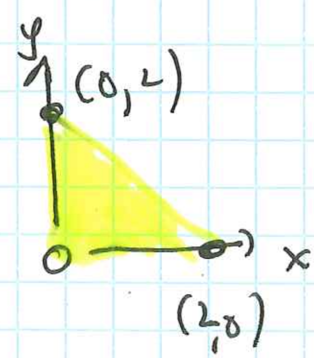
(2) $$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v),z(u,v))\cdot |\vec{r}_u \times \vec{r}_v|\,du\,dv$$



$$2x+2y+z=4$$



$$D = \{(x,y) : \ 0 \le x \le 2, \ 0 \le y \le 2-x\}$$

$$z = 4-2x-2y$$

$$\vec{r}(x,y) = \langle x, y, 4-2x-2y \rangle$$

$$\vec{r}_x = \frac{\partial \vec{r}}{\partial x} = \langle 1, 0, -2 \rangle$$

$$\vec{r}_y = \frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -2 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 2\hat{i} + 2\hat{j} + \hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{4+4+1} = 3$$

$$\iint_S xz \, dS =$$

$$\int_0^2 \left(\int_0^{2-x} \underbrace{x(4-2x-2y)}_{f(x,y,z) = xz \text{ along } S} \cdot \underbrace{3}_{|\vec{r}_x \times \vec{r}_y|} \, dy \right) dx$$