Math 213 - Functions of Several Variables

Peter Perry

September 18, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Functions of Several Variables

We'll begin to study functions such as

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$
$$g(x,y) = e^{-(x^2+y^2)}$$
$$h(x,y) = e^{-(x^2+y^2)} \cos\left(\sqrt{x^2+y^2}\right)$$

and address some basic questions:

- What are the domain and range of a function of several variables?
- How do you graph a function of several variables?
- What is the limit of a function of several variables at a point
- When is a function of several variables continuous?

Some Useful Notation

$\mathbb{N}$	The natural numbers $1, 2, 3, \ldots$
$\mathbb{R}^n$	Real n-dimensional space,where n is a natural number, i.e., the set of points $(x_1, \ldots, x_n)$ where x_i is a real number
$\in$	"is an element of"
$\notin$	"is not an element of"
$f : S \to T$	"The function f has domain S and range T"

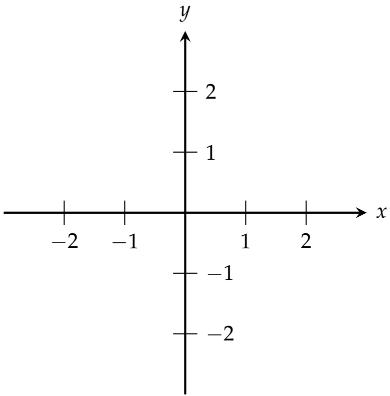
$[a, b]$ The interval $a \leq x \leq b$

$[a, b)$ The interval $a \leq x < b$

$(a, b]$ The interval $a < x \leq b$

(a, b) The interval $a < x < b$

Domain and Range



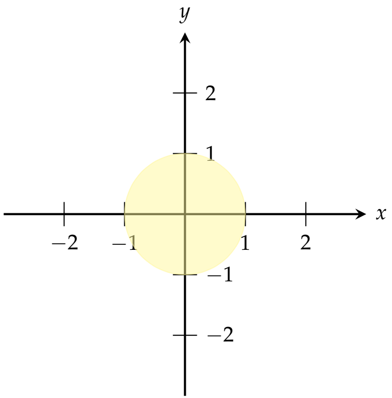
Find the domain of the function

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$

What is its range?

What is its graph?

Domain and Range



Find the domain of the function

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$f : S \to T$ where:

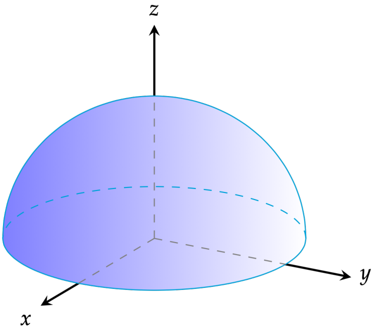
Domain:

$$S = \{(x, y) : x^2 + y^2 \leq 1\}$$

Range:

$$T = [0, 1]$$

Domain and Range



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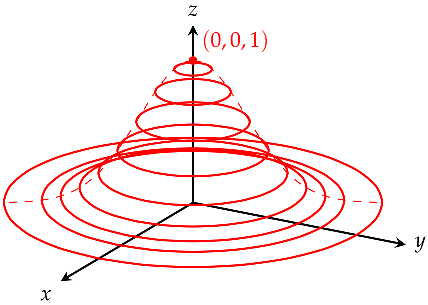


Find the domain of the function

$$g(x,y) = e^{-(x^2+y^2)}$$

What is its range?

What is its graph?

Domain: $\mathbb{R}^2$

Range: $(0, 1]$

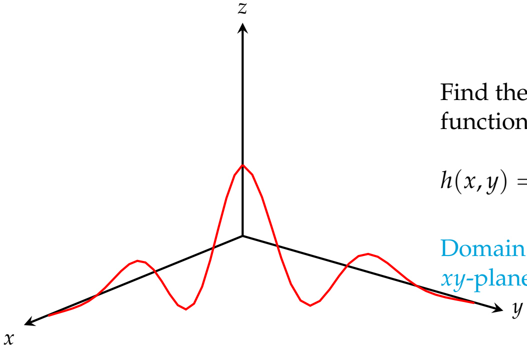
Domain and Range

Find the domain and graph of the function

$$h(x,y) = e^{-(x^2+y^2)} \cos\left(\sqrt{x^2+y^2}\right)$$

Domain: $\mathbb{R}^2$ (i.e., all of the xy-plane

Domain and Range



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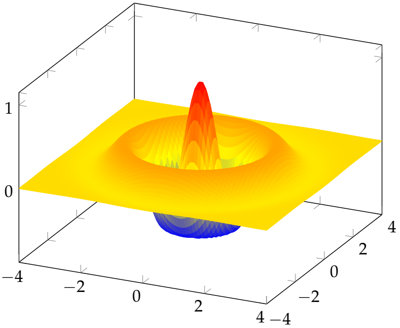
Domain and Range



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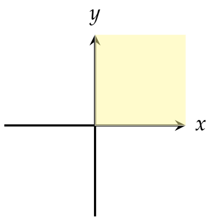
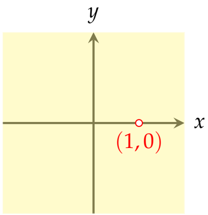
Domain and Range

Match these functions with their domains:

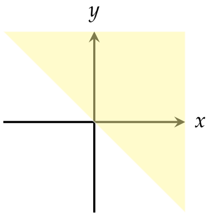
$$\ln(x+y) \qquad \frac{x^2}{(x-1)^2+y^2} \qquad \sqrt{x}+\sqrt{y}$$



Domain of $\sqrt{x}+\sqrt{y}$



Domain of $\frac{x^2}{(x-1)^2+y^2}$



Domain of $\ln(x+y)$

Limits

Suppose $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}$.

If $\mathbf{a}$ is a point in $\mathbb{R}^m$ and $f(\mathbf{x})$ is defined near $\mathbf{x} = \mathbf{a}$, we say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$$

if we can make $f(\mathbf{x})$ arbitrarily close to L by choosing $\mathbf{x}$ close enough to $\mathbf{a}$

Examples:

$$\lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = 1$$

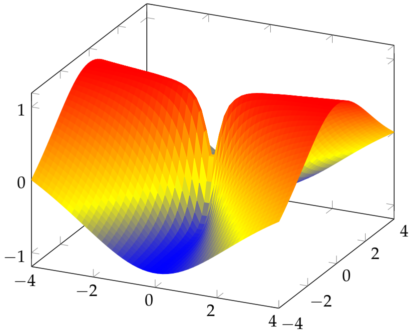
$$\lim_{(x,y)\to(\pi,\pi/2)} \cos(x)\sin(y) = -1$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} = 1$$

Limits

However, functions of two variables don't always have limits!

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$



Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Let's set

$$x = r\cos\theta, \quad y = r\sin\theta$$

What is $f(r\cos\theta, r\sin\theta)$?

$\cos^2\theta - \sin^2\theta$

What happens as $r \downarrow 0$ if $\theta = 0$?

1

What about if $\theta = \pi/4$?

0

Limit Laws

If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + g(\mathbf{x}) = L + M$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) - g(\mathbf{x}) = L - M$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \cdot g(\mathbf{x}) = LM$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/g(\mathbf{x}) = L/M \quad \text{if } M \neq 0$$

Compositions

If $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = \mathbf{b}$ and $\lim_{\mathbf{x}\to\mathbf{b}} f(\mathbf{x}) = L$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} f(g(\mathbf{x})) = L$$

If $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c$ and $\lim_{t\to c} f(t) = L$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} f(g(\mathbf{x}) = L$$

Continuity

Suppose that $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}$. We say that f is continuous at $\mathbf{x} = \mathbf{a}$ if

- $\mathbf{a}$ lies in the domain of f
- $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists
- $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$

If A is a set, we say that f is continuous on A if f is continuous at every $\mathbf{a} \in A$

- Polynomial and rational functions are continuous on their domains
- Roots and power functions are continuous on their domains
- Trig and inverse trig functions are continuous on their domains
- Exponential and logarithm functions are continuous on their domains
- Compositions of continuous functions are continuous

Reminders for the Week of September 18-22

- Exam Review Wednesday 9/22 in Class
- Exam 1 at 5:00 PM Wednesday 9/20
- No recitation on Thursday 9/21
- Exam scores should be posted by 5 PM on Thursday 9/21
- Unit B continues on Friday 9/22
- Your exams will be returned to you in Recitation on 9/26