Math 213 - Exam 1 Review

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September 20, 2023

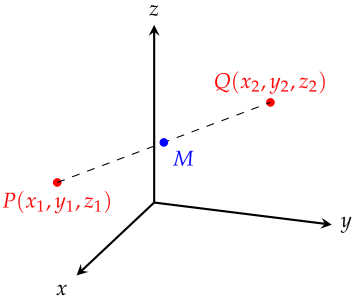
Reminders for the week of September 18-22

- Exam 1 is tonight at 5 PM
- Section 011 - CP 139
- Sections 012-014 - CP 153
- Bring your student ID and arrive at least 5 minutes early
- You are allowed one notebook page with definitions and formulas, but *no* solved problems

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces
- September 20 - Exam 1 Review

Cheat Sheet, Part I - Basics



If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$:

Distance formula:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Midpoint formula:

The midpoint between P and Q is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Cheat Sheet, Part II - Vector Products

Vector Products

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

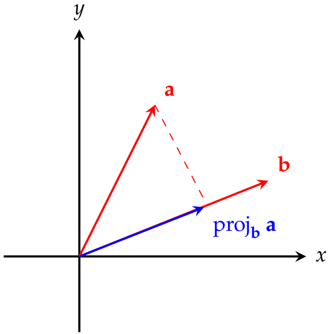
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Cheat Sheet, Part II - Vector Products

	Dot Product	**Cross Product**	**Scalar Triple Product**
Type	Scalar $\mathbf{a} \cdot \mathbf{b}$	Vector $\mathbf{a} \times \mathbf{b}$	Scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Magnitude	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin\theta$	
Symmetry	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	Antisymmetric
Direction	None!	Right-hand rule	None!
In Physics	Work	Torque	
In Geometry	Projection	Parallelogram Area	Parallelepiped Area
Zero If	$\mathbf{a} \perp \mathbf{b}$	$\mathbf{a} \parallel \mathbf{b}$	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ coplanar

Cheat Sheet, Part III: Projections



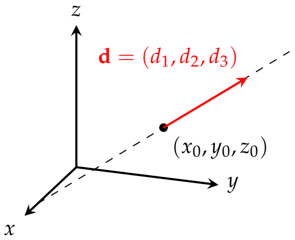
The projection of **a** onto **b** is given by

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|}$$

The first factor is a scalar, the *signed length* of the component of **a** in the **b** direction

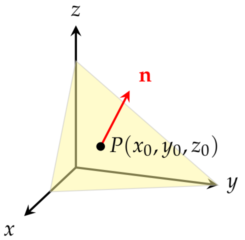
The second factor is a vector, a *unit vector* in the direction of **b**

Cheat Sheet, Part IV - Lines and Planes



Equation of a Line

$$\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\langle d_1, d_2, d_3\rangle$$

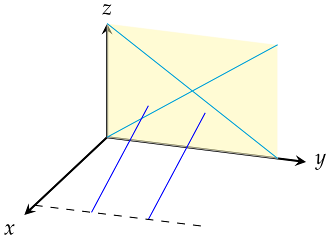


Equation of a Plane

$$ax + by + cz = d$$

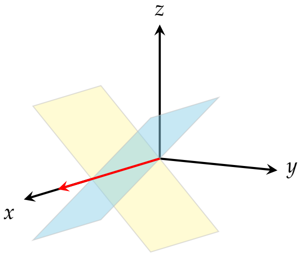
where $\mathbf{n} = \langle a, b, c\rangle$ is a vector normal to the plane, and d is determined by a point on the pane

Lines and Planes



Lines in three-dimensional space are either

- parallel,
- intersecting, or
- skew



Planes in three-dimensional space are either

- parallel, or
- intersecting

Lines and Planes

(1) Are the lines

$$x(s) = 1 + 2s \qquad x(t) = 5 + t$$
$$y(s) = -4 + s \qquad y(t) = 1 - t$$
$$z(s) = 8 + 2s \qquad z(t) = 8 - 3t$$

parallel, intersecting, or skew?

The lines aren't parallel because their displacement vectors $\langle 2, 1, 2\rangle$ and $\langle 1, -1, -3\rangle$ are not parallel

To see if they intersect, try to solve the equations

$$1 + 2s = 5 + t$$
$$-4 + s = 1 - t$$
$$8 + 2s = 8 - 3t$$

The first two equations are equivalent to

$$2s - t = 4$$
$$s + t = 5$$

$s = 3$, $t = 2$. This doesn't work in the third equation so the lines are skew.

Lines and Planes

(2) Are the planes

$$3x + y + 2z = 6$$
$$x + y + z = 3$$

parallel or intersecting? If intersecting, what is the equation for the line of intersection?

The normal vectors are $\mathbf{n}_1 = \langle 3, 1, 2 \rangle$ and $\mathbf{n}_2 = \langle 1, 1, 1 \rangle$ so the planes are not parallel. A displacement vector for the line of intersection is

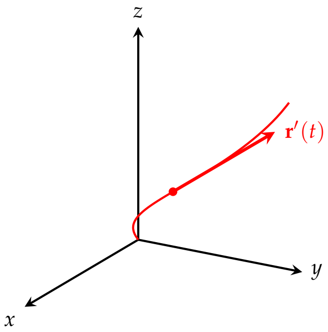
$$\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

We can find a point on the line by finding (x, y, z) that satisfy both equations. Subtracting twice the second equation from the first we get $x - y = 0$ so, for example, $(0, 0, 3)$ is a point in the intersection. So we get

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 0, 3 \rangle + t\langle -1, -1, 2 \rangle$$

Space Curves

A space curve is given by: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b$



$$\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle$$

The derivative

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

gives:

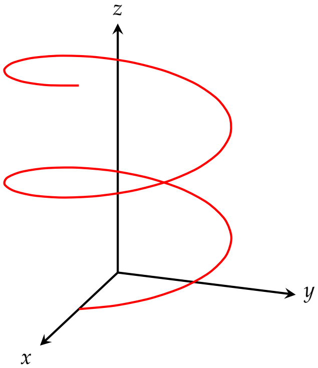
- The tangent vector $\mathbf{r}'(t)$
- The instantaneous speed $|\mathbf{r}'(t)|$
- The arc length.

$$L = \int_a^b |\mathbf{r}'(t)| \, dt$$

- The *arc length function*

$$s(t) = \int_a^t |\mathbf{r}'(t')| \, dt'$$

Reparametrization



Find the arc length function for the curve

$$\mathbf{r}(t) = \langle 3\cos(2t), 3\sin(2t), t \rangle$$

where $0 \leq t \leq 2\pi$ and reparametrize by arc length

Hint:

$$\mathbf{r}'(t) = \langle -6\sin(2t), 6\cos(2t), 1 \rangle$$

Since $\mathbf{r}'(t) = \langle -6\sin(2t), 6\cos(2t), 1 \rangle$ we get

$$|\mathbf{r}'(t)| = \sqrt{36\sin^2(2t) + 36\cos^2(2t) + 1} = \sqrt{37}$$

So

$$s(t) = \int_0^t \sqrt{37}\,dt = \sqrt{37}t.$$

Reparametrization

Given $s = \sqrt{37}t$ we can solve for t in terms of s:

$$t = \frac{s}{\sqrt{37}}$$

Subsituting the right-hand side for t in the equation

$$\mathbf{r}(t) = \langle 3\cos(2t), 3\sin(2t), t\rangle$$

we get

$$\mathbf{r}(s) = \langle 3\cos(2s/\sqrt{37}), 3\sin(2s/\sqrt{37}), s/\sqrt{37}\rangle$$

Integrals over Curves

If C is the curve

$$(x(t), y(t)), \quad a \leq t \leq b$$

then the integral of $f(x, y)$ over C is

$$\int_C f(x,y)\,ds = \int_a^b \underbrace{f(x(t),y(t))}_{f(x,y)}\ \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\,dt}_{ds}$$

If C is the curve

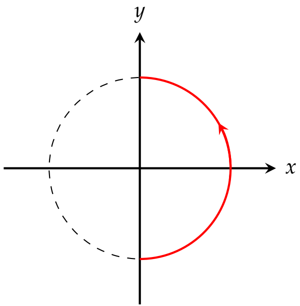
$$(x(t), y(t), z(t)), \quad a \leq t \leq b$$

then the integral of $f(x, y, z)$ over C is

$$\int_C f(x,y,z)\,ds = \int_a^b \underbrace{f(x(t),y(t),z(t))}_{f(x,y,z)}\ \underbrace{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}_{ds}$$

Integrals over Curves

Find $\int_C xy^4\,ds$ if C is the right half of the circle $x^2+y^2=16$ oriented in the counterclockwise direction.



First, the curve is parametrized as follows:

$$x(t) = 4\cos t$$
$$y(t) = 4\sin t$$

for $-\pi/2 \le t \le \pi/2$, and

$$x'(t) = -4\sin t$$
$$y'(t) = 4\cos t.$$

The speed is

$$\sqrt{x'(t)^2 + x'(t)^2} = 4$$

Now substitute

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos(t))(4\sin(t))^4\,4\,dt$$

Integrals Over Curves

Find $\int_C xyz\, ds$ if C is the curve

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle \quad 0 \le t \le 4\pi.$$

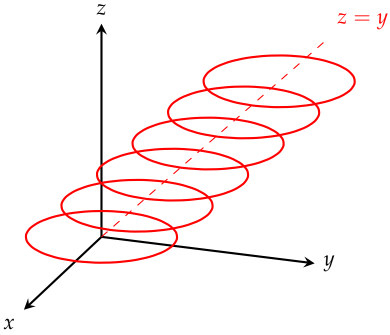
Since $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 3 \rangle$ you can check that

$$\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{10}.$$

Substituting for x,y,z and using $ds = \sqrt{10}\, dt$ we get

$$\int_C xyz\, ds = \int_0^{4\pi} t\cos(t)\sin(t)\sqrt{10}\, dt$$

Cylinders



$$x^2 + (y - z)^2 = 1$$

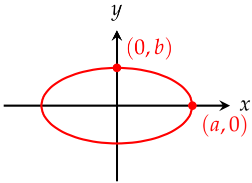
Trace in $z = h$:

$$x^2 + (y - h)^2 = 1$$

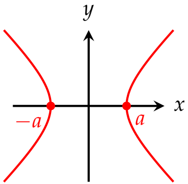
Circle of radius 1 and center $(0, h, h)$

Quadric Surfaces

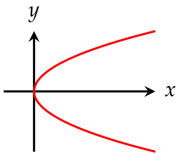
Traces of *quadric surfaces* are *conic sections*:



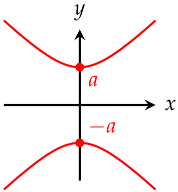
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



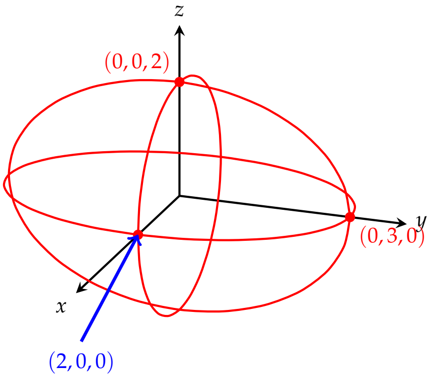
$$y^2 = 4px$$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Quadric Surfaces

Identify quadric surfaces from their equations by the *method of traces*



$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Traces for $x = h$:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - \frac{h^2}{4}$$

Traces for $y = h$:

$$\frac{x^2}{4} + \frac{z^2}{4} = 1 - \frac{h^2}{9}$$

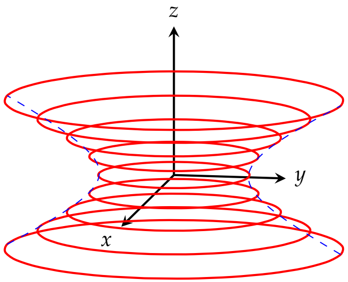
Traces for $z = h$:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 - \frac{h^2}{4}$$

You can take $h = 0$ and see that the traces of this surface in the xy, xz, and yz planes are all ellipses, and you can find where these ellipses intersect the x, y, and z axes.

Quadric Surfaces

Identify quadric surfaces from their equations by the *method of traces*



$$\frac{x^2}{4} + \frac{y^2}{4} - z^2 = 1$$

Traces for $z = h$:

$$\frac{x^2}{4} + \frac{y^2}{4} = 1 + h^2$$

Traces for $x = 0$:

$$\frac{y^2}{4} - z^2 = 1$$

The traces of this surface in planes $z = h$ are circles:

$$x^2 + y^2 = 4(1 + h^2)$$

so the circles have center $(0, 0)$ and radius $2\sqrt{1 + h^2}$. The traces in the yz plane are hyperbolas (blue dashed lines).