Math 213 - Partial Derivatives

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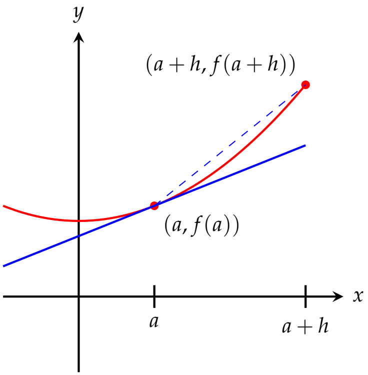
September 22, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Derivative Review



If $f(x)$ is a function of one variable, its derivative at $x = a$ is

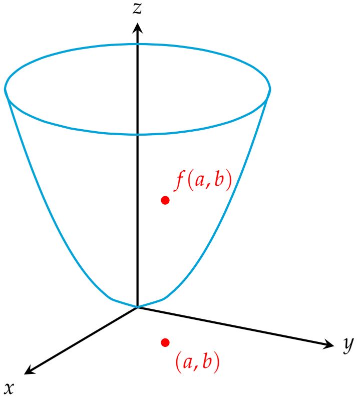
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative is the slope of the tangent line to the graph of $f(x)$ at $x = a$

The derivative is also the instantaneous rate of change of $f(x)$ at $x = a$

If $f(x)$ is differentiable at $x = a$, $f(x)$ is also continuous at $x = a$

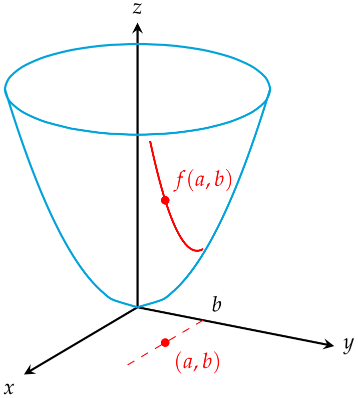
Partial Derivatives



Suppose that $f(x, y) = x^2 + y^2$.

There are now *two* rates of change of $f(x, y)$ at (a, b):

Partial Derivatives



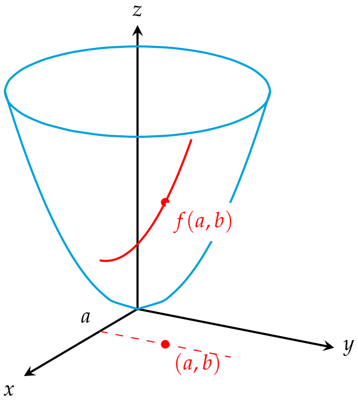
Suppose that $f(x, y) = x^2 + y^2$.

There are now *two* rates of change of $f(x, y)$ at (a, b):

- The rate of change of f with respect to x at (a, b) for *fixed* $y = b$, denoted

$$\frac{\partial f}{\partial x}(a, b)$$

Partial Derivatives



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- The rate of change of f with respect to y at (a,b) for *fixed* $x = a$, denoted

$$\frac{\partial f}{\partial y}(a,b)$$

Partial Derivatives

$$f(x, y) = x^2 + y^2$$

Let's fix $y = b$ and define a function of *one* variable:

$$B(x) = f(x, b) = x^2 + b^2$$

Then

$$\frac{\partial f}{\partial x}(a, b) = B'(a)$$

Let's fix $x = a$ and define a function of *one* variable:

$$C(y) = a^2 + y^2$$

Then

$$\frac{\partial f}{\partial y}(a, b) = C'(b)$$

Partial Derivatives How-To

$$f(x,y) = x^2 + y^2$$

We found:

$$\frac{\partial f}{\partial x}(a,b) = 2a, \quad \frac{\partial f}{\partial y}(a,b) = 2b$$

For any (x,y):

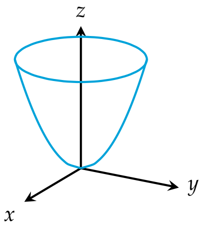
$$\frac{\partial f}{\partial x}(x,y) = 2x, \quad \frac{\partial f}{\partial y}(x,y) = 2y$$

Moral:

- To find $\partial f / \partial x$, treat y as a constant and differentiate with respect to x
- To find $\partial f / \partial y$, treat x as a constant and differentiate with respect to y

Functions to Remember

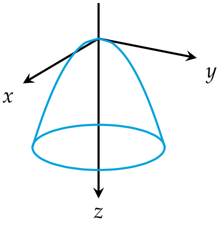
$f(x,y) = x^2 + y^2$



$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

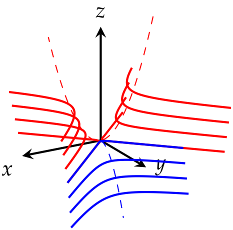
$f(x,y) = -(x^2 + y^2)$



$$\frac{\partial f}{\partial x} = -2x$$

$$\frac{\partial f}{\partial y} = -2y$$

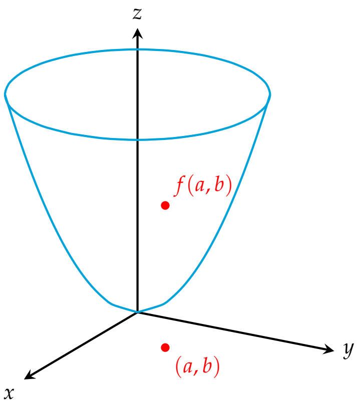
$f(x,y) = x^2 - y^2$



$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = -2y$$

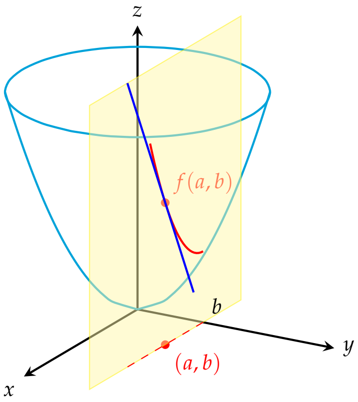
Geometric Interpretation



Suppose that $f(x, y) = x^2 + y^2$.

The derivatives $(\partial f/\partial x)(a, b)$ and $(\partial f/\partial y)(a, b)$ have the following geometric interpretations:

Geometric Interpretation

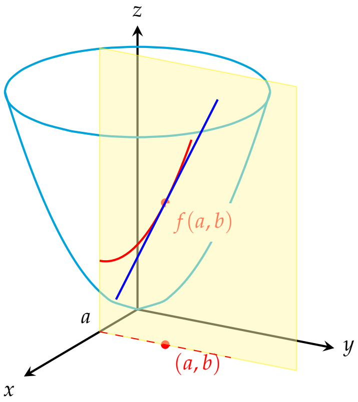


Suppose that $f(x, y) = x^2 + y^2$.

The derivatives $(\partial f / \partial x)(a, b)$ and $(\partial f / \partial y)(a, b)$ have the following geometric interpretations:

- $\dfrac{\partial f}{\partial x}(a, b)$ is the slope of the tangent line to the graph of $B(x) = f(x, b)$ at $x = a$

Geometric Interpretation



Suppose that $f(x,y) = x^2 + y^2$.

The derivatives $(\partial f/\partial x)(a,b)$ and $(\partial f/\partial y)(a,b)$ have the following geometric interpretations:

- $\dfrac{\partial f}{\partial x}(a,b)$ is the slope of the tangent line to the graph of $B(x) = f(x,b)$ at $x = a$

- $\dfrac{\partial f}{\partial y}(a,b)$ is the slope of the tangent line to the graph of $C(y) = f(a,y)$ at $y = b$

Puzzler # 1

$\dfrac{\partial f}{\partial x}$ - derivative with respect to x, y held constant

$\dfrac{\partial f}{\partial y}$ - derivative with respect to y, x held constant

Find $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y}$ for the following functions:

- $f(x,y) = x^3 - 3xy^2 + y^2$
- $f(x,y) = x^2 \cos(y)$
- $f(x,y) = e^{-(x^2+y^2)}$
- $f(x,y) = \dfrac{x^2}{x-y}$

Puzzler #2

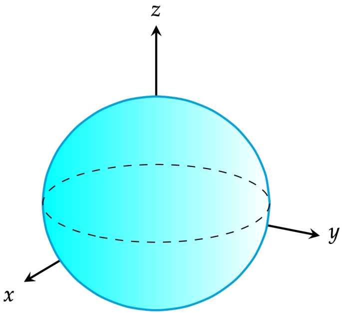
Now consider functions of three variables $f(x, y, z)$ which have partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

Find the partial derivatives of $f(x, y, z) = x^2y^2 + xyz$ and evaluate them at $(1, 0, 0)$

Implicit Differentiation

Suppose that $x^2 + y^2 + z^2 = 1$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$



Note that $z = \pm\sqrt{1 - x^2 - y^2}$ but avoid using this fact!

A Cautionary Tale

Let

$$f(x,y) = \begin{cases} \dfrac{x^2}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$$

Find the partials at $(0,0)$ from the definition of the derivative.

If $f(x,y)$ continuous at $(0,0)$?

Reminders for the Week of September 25-29

- Homework B1 on Limits due 9/25 at 11:59 PM
- Homework B2 on Partial Derivatives due 9/27 at 11:59 PM
- Quiz # 4 on limits, partial derivatives due 9/28 at 11:59 PM
- Homework B3 on the Chain Rule due 9/29 at 11:59 PM