Math 213 - Higher-Order Partial Derivatives

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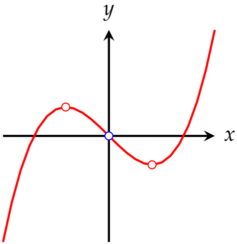
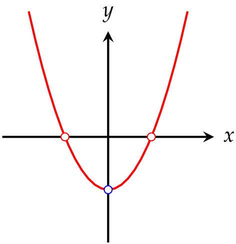
Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

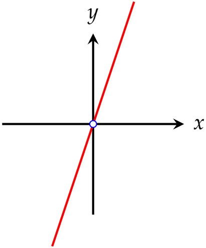
What Second Derivatives are Good For



$f(x) = x^3 - x$



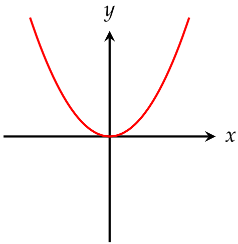
$f'(x) = 3x^2 - 1$



$f''(x) = 6x$

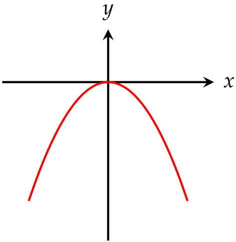
Can you locate the maximum, minimum, and inflection point for $f(x)$?

What Second Derivatives are Good For



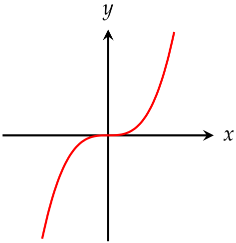
$$f(x) = x^2$$
$$f'(0) = 0$$
$$f''(0) = 2$$

Absolute Mininum at $x = 0$



$$f(x) = -x^2$$
$$f'(0) = 0$$
$$f''(0) = -2$$

Absolute Maximum at $x = 0$



$$f(x) = x^3$$
$$f'(0) = 0$$
$$f''(0) = 0$$

Neither at $x = 0$

Higher-Order Partial Derivatives

Suppose that $f(x, y) = x^3 + xy^2$. The first derivatives are:

$$\frac{\partial f}{\partial x}(x, y) = 3x^2 + 2xy, \qquad \frac{\partial f}{\partial y}(x, y) = 2xy$$

Let's find the second derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) =$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) =$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) =$$

Partial Derivative Notation

Here are shorthand ways of denoting second partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

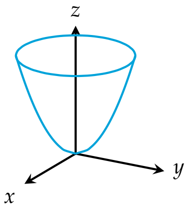
Note:

$\frac{\partial^2 f}{\partial y \partial x}$ means differentiate in x first, then in y

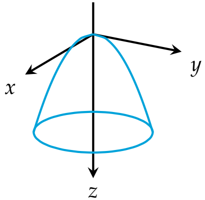
f_{xy} means differentiate in x first, then y

Functions to Remember

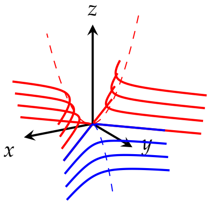
$f(x,y) = x^2 + y^2$



$f(x,y) = -(x^2 + y^2)$



$f(x,y) = x^2 - y^2$



$f_x = 2x$

$f_y = 2y$

$f_{xx} = 2, \quad f_{yy} = 2$

$f_{xy} = f_{yx} = 0$

$f_x = -2x$

$f_y = -2y$

$f_{xx} = -2, \quad f_{yy} = -2$

$f_{xy} = f_{yx} = 0$

$f_x = 2x$

$f_y = -2y$

$f_{xx} = 2, \quad f_{yy} = -2$

$f_{xy} = f_{yx} = 0$

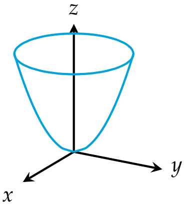
Derivatives to Remember - Sneak Preview

If $f(x, y)$ is a function of two variables:

- The vector
$$(\nabla f)(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$
is called the *gradient vector* - it is the real "first derivative." It vanishes at critical points of $f(x, y)$. It points in the direction of greatest change of f
- The matrix
$$\text{Hess}(f)(x, y) = \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}$$
is called the *Hessian matrix* and is the real "second derivative" for a function of two variables. It is used in the second derivative test for maxima and minima.

The Gradient and the Hessian - Sneak Preview

$$f(x,y) = x^2 + y^2$$

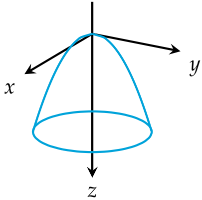


$$(\nabla f)(x,y) = \langle 2x, 2y \rangle$$

$$\text{Hess}(f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Absolute Minimum at $(0,0)$

$$f(x,y) = -(x^2 + y^2)$$

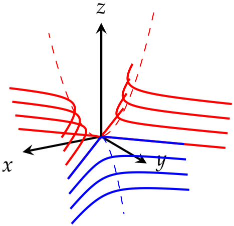


$$(\nabla f)(x,y) = \langle -2x, -2y \rangle$$

$$\text{Hess}(f)(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Absolute Maximum at $(0,0)$

$$f(x,y) = x^2 - y^2$$



$$(\nabla f)(x,y) = \langle 2x, -2y \rangle$$

$$\text{Hess}(f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Neither at $(0,0)$

Functions of Three Variables

Let

$$f(x, y, z) = e^{-\sqrt{2}z}\cos(x)\sin(y)$$

Find

$$f_{xx} + f_{yy} + f_{zz}$$

$$f_{xx}(x, y, z) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$f_{yy}(x, y, z) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$f_{zz}(x, y, z) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Puzzler #1 - Light

Let $u(x, y) = \sin(x + y)$. Show that $u(x, y)$ solves the differential equation

$$u_{xx} - u_{yy} = 0$$

by finding the derivatives u_{xx} and u_{yy}.

$u_x =$ ______________ $u_y =$ ______________

$u_{xx} =$ ______________ $u_{yy} =$ ______________

$u_{xx} - u_{yy} =$ ____________________

Puzzler #2 - Heat

Let $u(x,t) = t^{-1/2}e^{-x^2/4t}$. Show that $u(x,t)$ solves the equation

$$u_t = u_{xx}$$

by finding the derivatives u_t and u_{xx}.

$$u_t = -\frac{1}{2}t^{-\frac{3}{2}}e^{-x^2/4t} + \frac{1}{4}t^{-\frac{5}{2}}x^2e^{-x^2/4t}$$

$$u_x = -\frac{1}{2}t^{-\frac{3}{2}}xe^{-x^2/4t}$$

$$u_{xx} = \underline{\hspace{10cm}}$$

Puzzler #3 - Patterns

Let $f(x,y) = e^{\alpha x + \beta y}$ where α and β are constants. Can you find a formula for

$$\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(x,y)$$

Hint: Try computing the first few derivatives and see if you see a pattern emerging.

$$\frac{\partial f}{\partial x}(x,y) = \underline{\hspace{4cm}} \qquad \frac{\partial f}{\partial y}(x,y) = \underline{\hspace{4cm}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \underline{\hspace{4cm}} \qquad \frac{\partial^2 f}{\partial y \partial x}(x,y) = \underline{\hspace{4cm}}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \underline{\hspace{4cm}} \qquad \frac{\partial^2}{\partial y^2}(x,y) = \underline{\hspace{4cm}}$$

More Patterns - Clairaut's Theorem

Can you find the pattern in this picture?

$$\begin{aligned} u(x,y) &= x^2y + xy \\ u_x(x,y) &= 2xy + y \\ u_y(x,y) &= x^2 + x \\ u_{xy}(x,y) &= 2x + 1 \\ u_{yx}(x,y) &= 2x + 1 \end{aligned}$$

$$\begin{aligned} v(x,y) &= \sin(x)\cos(y) \\ v_x(x,y) &= \cos(x)\cos(y) \\ v_y(x,y) &= -\sin(x)\sin(y) \\ v_{xy}(x,y) &= -\cos(x)\sin(y) \\ v_{yx}(x,y) &= -\cos(x)\sin(y) \end{aligned}$$

Clairaut's Theorem

Theorem (Clairaut) If the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}(x, y)$ and $\frac{\partial^2 f}{\partial y \partial x}(x, y)$ exist and are continuous at (x_0, y_0), then

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

This theorem asserts the "equality of mixed partials"

History Break - Clairaut and du Chatelet

Alexis Claude Clairaut (1713-1765) was a French mathematician who worked to prove Newton's conjecture that the earth is an oblate spheroid. He also helped the Marquise du Chatelet (1706-1749) in her celebrated translation of Newton's *Principia Mathematica* from Latin into French.

Alexis Claude Clairaut

Portrait Courtesy of MacTutor History of Mathematics

Emilie Chatelet

Portrait by Quentin Maurice Latour, courtesy of Wikipida Commons

Counterexample (Time Permitting)

The function

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

has first derivatives

$$f_x(x,y) = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{4xy^2}{(x^2+y^2)^2}$$

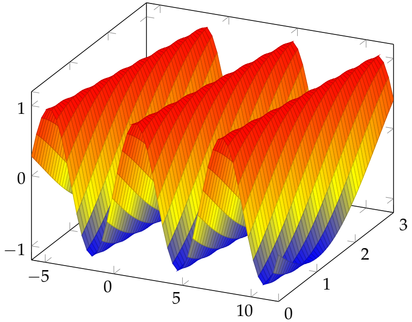
$$f_y(x,y) = x\frac{x^2-y^2}{x^2+y^2} - xy\frac{4yx^2}{(x^2+y^2)^2}$$

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ by using the formulas

$$f_{xy}(0,0) = \frac{d}{dy}f_x(0,y), \qquad f_{yx}(0,0) = \frac{d}{dx}f_y(x,0)$$

and the definition of the derivative as a limit.

More Light



The equation

$$u_{xx} = u_{tt}$$

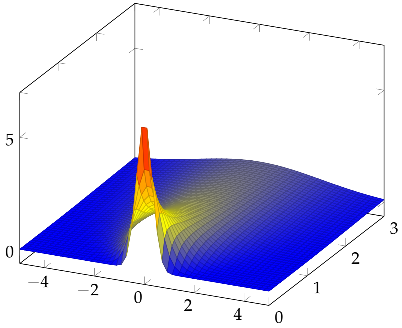
for a function $u(x, t)$ describes wave motion. The solution

$$u(x, t) = \sin(x - t)$$

describes a rightward moving wave. The graph shows

$$0 \leq t \leq 3, \quad -5 \leq x \leq 10$$

More Heat



The equation

$$u_t = u_{xx}$$

describes the temperature in a rod at time t (x is the distance from the origin along the rod). The function

$$u(x,t) = t^{-\frac{1}{2}} e^{-x^2/4t}$$

describes the temperature of a rod at time t if, at time 0, the point $x = 0$ on the rod is a “hotspot”

More Homework

- Homework B1 on Limits due 9/25 at 11:59 PM
- Recitation on partial derivatives, higher-order derivatives, 9/26
- Homework B2 on Partial Derivatives due 9/27 at 11:59 PM
- Recitation on the chain rule, 9/28
- Quiz # 4 on limits, partial derivatives due 9/28 at 11:59 PM
- Homework B3 on the Chain Rule due 9/29 at 11:59 PM