Math 213 - The Chain Rule

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September 27, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

The Chain Rule for Functions of One Variable

If $y = f(u), u = g(x)$, and $F(x) = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dF}{dx}(x) = f'(g(x)) \cdot g'(x)$$

Find:

1. dy/dx if $y = (x^2 + 1)^5$
2. dy/dx if $y = \cos(\tan(x))$
3. dy/dx if $y = e^{x^2+5x}$

The Chain Rule for Functions of Two Variables - I

Chain Rule: If $f = f(x,y)$, $x = x(s,t)$, $y = y(s,t)$, and $F(s,t) = f(x(s,t),y(s,t))$ then

$$\frac{\partial F}{\partial s}(s,t) = \frac{\partial f}{\partial x}(x(s,t),y(s,t))\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x(s,t),y(s,t))\,\frac{\partial y}{\partial s}(s,t)$$

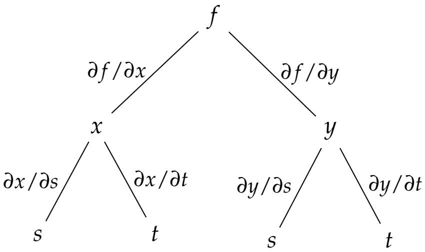
$$\frac{\partial F}{\partial t}(s,t) = \frac{\partial f}{\partial x}(x(s,t),y(s,t)\,\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x(s,t),y(s,t)\,\frac{\partial y}{\partial t}(s,t)$$

Find $\partial F/\partial s$ and $\partial F/\partial t$ if

$$\begin{aligned} f(x,y) &= x^2 + 2xy \\ x(s,t) &= 2s + 3t \\ y(s,t) &= s^2 + t^2 \end{aligned}$$

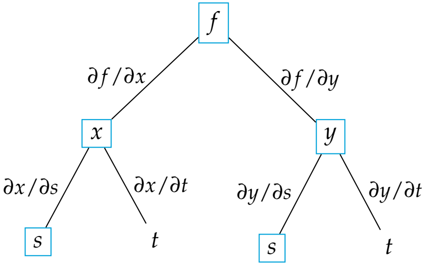
and $F(s,t) = f(x(s,t),y(s,t))$

The Chain Rule Tree



You can remember the chain rule using the tree diagram at left:

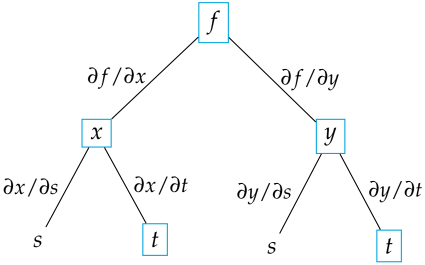
The Chain Rule Tree



You can remember the chain rule using the tree diagram at left:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

The Chain Rule Tree



You can remember the chain rule using the tree diagram at left:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Let's Compute!

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

If

$$f(x,y) = x^2y + xy^3, \quad x = s\cos t, \quad y = s\sin t,$$

and

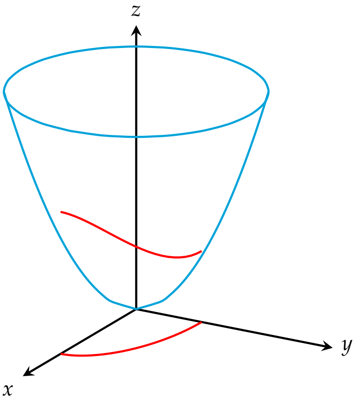
$$F(s,t) = f(x(s,t), y(s,t)),$$

find $\partial F/\partial r$ and $\partial F/\partial t$.

$$\frac{\partial f}{\partial x} = \qquad \frac{\partial x}{\partial s} = \qquad \frac{\partial x}{\partial t} =$$

$$\frac{\partial f}{\partial y} = \qquad \frac{\partial y}{\partial s} = \qquad \frac{\partial y}{\partial t} =$$

How a Function Changes along a Curve



Suppose that

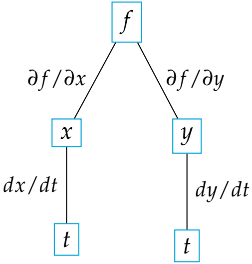
$$x(t) = 2\cos(t), \quad y(t) = \sin(t)$$

If

$$f(x, y) = x^2 + y^2,$$

find the rate of change of $f(x(t), y(t))$ with respect to t

How a Function Changes along a Curve



Given $f(x,y)$ and functions $x(t), y(t)$, if $F(t) = f(x(t), y(t))$, then

$$\frac{d}{dt}F(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Implicit Differentiation

Suppose that z is defined implicitly as a function of x, y via

$$x^2y + 2yz = 1$$

Find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation.

Numerical Differentiation

Suppose $F(s,t) = f(x(s,t), y(s,t))$ and

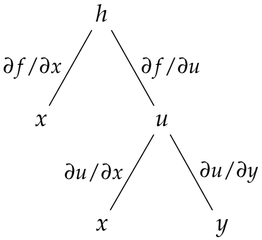
$$x(1,2) = 3, \quad y(1,2) = 4, \quad (\partial x/\partial s)(1,2) = 5, \quad (\partial y/\partial s)(1,2) = -4$$

Suppose that

$$f_x(3,4) = 0, \quad f_y(3,4) = 10.$$

Find $(\partial F)(\partial s)(1,2)$.

Up A Tree



Suppose

$$h(x, y) = f(x, u(x, y)).$$

Find $\partial h/\partial x$ and $\partial f/\partial y$ in terms of derivatives of f and u.

Reminders for the Week of September 25-29

- Homework B1 on Limits due tonight at 11:59 PM
- Homework B2 on Partial Derivatives due 9/27 at 11:59 PM
- Quiz # 4 on limits, partial derivatives due 9/28 at 11:59 PM
- Homework B3 on the Chain Rule due 9/29 at 11:59 PM