Math 213 - Tangent Planes and Normal Lines

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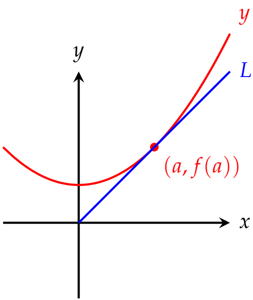
September 29, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

One-Variable Calculus Review

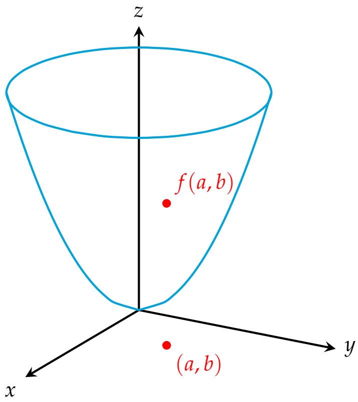


The derivative of f at $x = a$ defines a *tangent line* to the graph of $y = f(x)$

The derivative also defines a *linear approximation* $L(x)$ to $f(x)$ near $x = a$ given by

$$L(x) = f(a) + f'(a)(x - a)$$

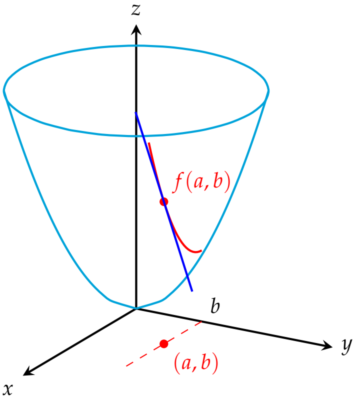
Tangent Plane to the Graph of $z = f(x, y)$



We'll see how to find

- the *tangent plane* to the graph of $z = f(x, y)$ at $(x, y) = (a, b)$
- a linear approximation $L(x, y)$ to $f(x, y)$ near $(x, y) = (a, b)$.

Tangent Plane to the Graph of $z = f(x, y)$



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- a linear approximation $L(x, y)$ to $f(x, y)$ near $(x, y) = (a, b)$.

Step 1: Find a tangent vector to the curve $z = f(x, b)$ at $(x, y) = (a, b)$

The tangent line has equation

$$z = f(a, b) + \partial f / \partial x(a, b) \cdot (x - a)$$

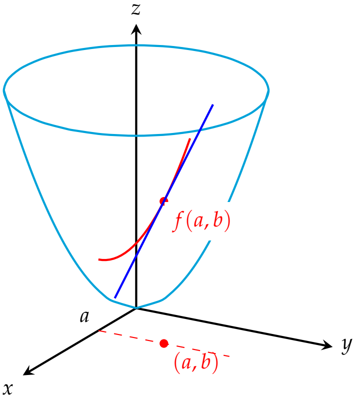
or

$$\langle x, y, z \rangle = \langle a, b, f(a, b) \rangle + t \langle 1, 0, \frac{\partial f}{\partial x}(a, b) \rangle$$

so

$$\mathbf{d}_x = \left\langle 1, 0, \frac{\partial f}{\partial x}(a, b) \right\rangle$$

Tangent Plane to the Graph of $z = f(x,y)$



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- the *tangent plane* to the graph of $z = f(x,y)$ at $(x,y) = (a,b)$
- a linear approximation $L(x,y)$ to $f(x,y)$ near $(x,y) = (a,b)$.

Step 2: Find a tangent vector to the curve $z = f(a,y)$ at $(x,y) = (a,b)$

The tangent line has equation

$$z = f(a,b) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

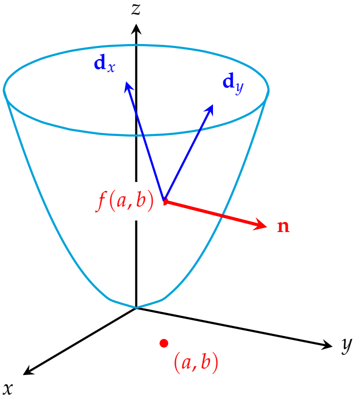
or

$$\langle x,y,z\rangle = \langle a,b,f(a,b)\rangle + t\langle 0,1,\frac{\partial f}{\partial y}(a,b)\rangle$$

so

$$\mathbf{d}_y = \left\langle 0,1,\frac{\partial f}{\partial y}(a,b)\right\rangle$$

Tangent Plane to the Graph of $z = f(x, y)$



The vectors

$$\mathbf{d}_x = \left\langle 1, 0, \frac{\partial f}{\partial x}(a,b) \right\rangle, \quad \mathbf{d}_y = \left\langle 0, 1, \frac{\partial f}{\partial y}(a,b) \right\rangle$$

span the *tangent plane* to the graph of $f(x, y)$ at $(a, b, f(a, b))$

How do we find its equation?

- A *normal* to the plane is $\mathbf{n} = \mathbf{d}_x \times \mathbf{d}_y$
- A *point* on the plane is $(a, b, f(a, b))$

Tangent Plane to the Graph of $z = f(x,y)$

Two tangent vectors to the plane are

$$\mathbf{d}_x = \left\langle 1, 0, \frac{\partial f}{\partial x}(a,b) \right\rangle$$

$$\mathbf{d}_y = \left\langle 0, 1, \frac{\partial f}{\partial y}(a,b) \right\rangle$$

So the normal vector is

$$\mathbf{n} = \left\langle -\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1 \right\rangle$$

The equation of the plane is:

$$-\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + (z - f(a,b)) = 0$$

Tangent Plane to the Graph of $z = f(x,y)$ at (a,b)

$$-\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + (z - f(a,b)) = 0$$

Solve for z to get

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

This equation defines the tangent plane and also gives a linear approximation to $f(x,y)$ near $(x,y) = (a,b)$:

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Puzzler #1

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Find the equation of the tangent plane and the linear approximation to $f(x, y) = x^3 e^y$ at $(1, 0)$.

Normal Line to the Tangent Plane

The *tangent plane* to the graph of $f(x,y)$ at $(a,b,f(a,b))$ has normal

$$\mathbf{n} = \left\langle -\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1 \right\rangle$$

and contains the point

$$(a,b,f(a,b))$$

The *normal line* to the tangent plane is

$$\mathbf{r}(t) = \underbrace{\langle a,b,f(a,b)\rangle}_{\text{point}} + t \underbrace{\left\langle -\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1 \right\rangle}_{\text{displacement vector}}$$

Find the equation of the normal line through the tangent plane to the graph of $f(x,y) = x \cos y$ at $(a,b) = (1,0)$.

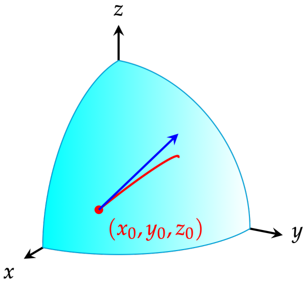
Puzzler #2

Normal line to graph of $z = f(x, y)$ at $(x, y) = (a, b)$:

$$\mathbf{r}(t) = \langle a, b, f(a, b)\rangle +$$
$$t\left\langle -\frac{\partial f}{\partial x}(a, b), -\frac{\partial f}{\partial y}(a, b), 1\right\rangle$$

Find the normal line to the graph of $f(x, y) = x^2 - 2y^2$ at $(x, y) = (2, 1)$

Tangent Plane $G(x,y,z) = 0$



Suppose that G is a function of three variables so the set of (x,y,z) with

$$G(x,y,z) = 0$$

is a surface. If (x_0, y_0, z_0) is a point on the surface, what is the equation of the tangent plane at (x_0, y_0, z_0)?

We will study curves $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ with

$$\mathbf{r}(0) = \langle x_0, y_0, z_0 \rangle.$$

and find a vector that is always normal to the tangent vector $\mathbf{r}'(0)$.

Tangent Plane to $G(x,y,z) = 0$

To find the tangent plane to a surface $G(x,y,z) = 0$ we need to find a normal vector to the surface. A curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ contained in the surface satisfies $G(x(t), y(t), z(t)) = 0$. Now differentiate with respect to t:

$$\frac{\partial G}{\partial x}(x(t),y(t),z(t))x'(t) + \frac{\partial G}{\partial y}(x(t),y(t),z(t))y'(t) + \frac{\partial G}{\partial z}(x(t),y(t),z(t))z'(t) = 0.$$

Now set $t = 0$ with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$:

$$\frac{\partial G}{\partial x}(x_0,y_0,z_0)x'(0) + \frac{\partial G}{\partial y}(x_0,y_0,z_0)y'(0) + \frac{\partial G}{\partial z}(x_0,y_0,z_0)z'(0) = 0.$$

If we denote

$$(\nabla G)(x_0,y_0,z_0) = \left\langle \frac{\partial G}{\partial x}(x_0,y_0,z_0), \frac{\partial G}{\partial y}(x_0,y_0,z_0), \frac{\partial G}{\partial z}(x(t),y(t),z(t)) \right\rangle$$

then

$$(\nabla G)(x_0,y_0,z_0) \cdot \mathbf{r}'(0) = 0.$$

Tangent Plane to $G(x, y, z) = 0$

The vector

$$(\nabla G)(x_0, y_0, z_0) = \left\langle \frac{\partial G}{\partial x}(x_0, y_0, z_0), \frac{\partial G}{\partial y}(x_0, y_0, z_0), \frac{\partial G}{\partial z}(x(t), y(t), z(t)) \right\rangle$$

is called the *gradient* of G at (x_0, y_0, z_0). If $\mathbf{r}(t)$ is any curve on the surface with $\mathbf{r}(0) = (x_0, y_0, z_0)$, then

$$(\nabla G)(x_0, y_0, z_0) \cdot \mathbf{r}'(0) = 0$$

Inotherwords, the gradient of G at (x_0, y_0, z_0) is a normal to the tangent plane!

Tangent Plane to $G(x, y, z) = 0$

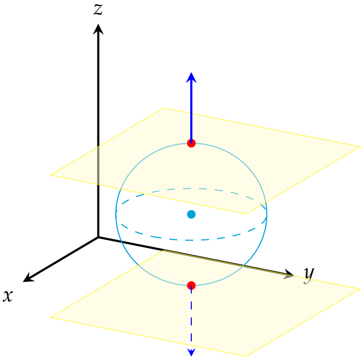
$(\nabla G)(x_0, y_0, z_0)$ is normal to $G(x, y, z) = 0$ at (x_0, y_0, z_0)

Find the tangent plane and normal line to the surface $x^2 + y^2 - z^2 = 4$ at the point $(2, -3, 3)$.

$(\nabla G)(x, y, z) =$ ______________________

$(\nabla G)(2, -3, 3) =$ ______________________

Maxima and Minima, Part I



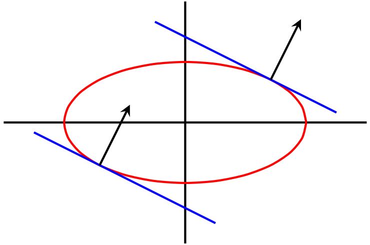
Find the maximum and minimum values of z for the surface

$$x^2 + y^2 + z^2 - 2x - 4y - 2z + 4 = 0$$

$$(\nabla G)(x,y,z) = \langle 2x-2, 2y-4, 2z-2 \rangle$$

Where is the gradient vertical?

Maxima and Minima, Part II



Find the points on the surface

$$x^2 + 2y^2 + 3z^2 = 72$$

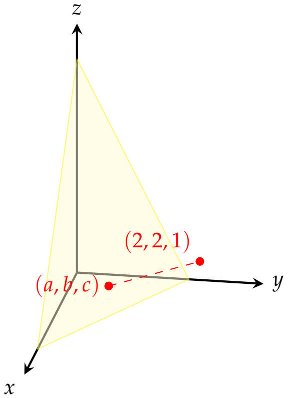
with the largest and smallest values of $x + y + 3z$.

What is the normal to the surface at any point (x, y, z)?

What is a normal to the plane $x + y + 3z = c$ for any c?

Problem courtesy of CLP 3, §2.5

Distance from a Point to a Plane

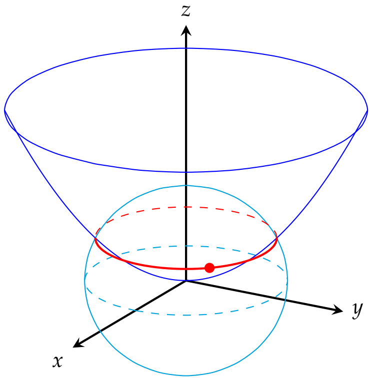


Find the distance from the point $(2, 2, 1)$ to the surface $G(x, y, z) = 0$ where

$$G(x, y, z) = x + 2y + z - 3$$

Use the fact that $\langle 2 - a, 2 - b, 1 - c \rangle$ points in the direction of $(\nabla G)(a, b, c)$

Tangent to a Curve of Intersection



Find a tangent vector the curve of intersection of the surfaces

$$x^2 + y^2 + z^2 = 5$$

and

$$x^2 + y^2 = 4z$$

at the point $(\sqrt{2}, \sqrt{2}, 1)$

Hint: Use the normals at $(\sqrt{2}, \sqrt{2}, 1)$ to the surfaces

Reminders for the Weeks of September 25-29 and October 2-6

- Homework B3 on the Chain Rule due tonight at 11:59 PM
- Homework B4 on Tangent Planes and Linear Approximation due 10/4 at 11:59 PM
- Quiz #5 on Higher-order derivatives, the chain rule, tangent planes and linear approximation due 10/5 at 11:59 PM